Interaction of Neutrons with Matter

Atom Density

- Also called ***number density***.
- Is the Number of Atoms per Unit Volume
- Connection with (mass) density
  - n = # of atoms in volume V
  - M = atomic weight of each atom
  - N = Atom density

$$\left.\begin{array}{l} N = \dfrac{n}{V} \\[2ex] \rho = \dfrac{m}{V} \end{array}\right\} \Rightarrow \rho = \frac{m}{V} = \frac{nM}{V} = M\frac{n}{V} = MN$$

Neutrons

- Interact with nuclei via nuclear forces, since they have no charge, hence they cannot interact electrostatically with electrons
- Possible reactions
  - Elastic scattering
  - Inelastic Scattering
  - radiative capture (absorption)
  - (n, 2n)
  - fission

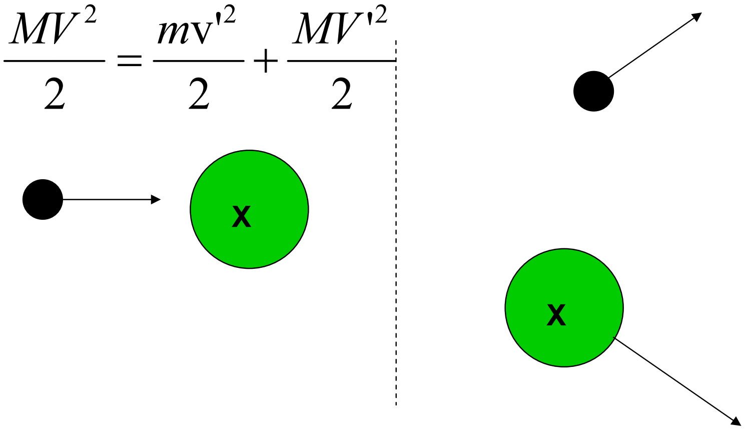
Neutron Elastic Scattering

$$n + {}^{A}_{Z}X \rightarrow n + {}^{A}_{Z}X$$

Kinetic energy is conserved

$$KE_n + KE_X = KE'_n + KE'_X$$

$$\frac{m\mathrm{v}^2}{2} + \frac{MV^2}{2} = \frac{m\mathrm{v}'^2}{2} + \frac{MV'^2}{2}$$



The incident neutron is slowed down by elastic scattering

Some of its kinetic energy is transferred to the target nucleus

Energy Loss in Elastic Scattering Collisions - Moderation

$$\overline{E}'=\frac{1}{2}\left[1+\left(\frac{A-1}{A+1}\right)^2\right]E=\frac{1+\alpha}{2}E$$

Scattering of heavy nucleus (^{235}U)- small energy loss (poor moderator)

$$\overline{E}'=\frac{1}{2}\left[1+\left(\frac{234}{236}\right)^2\right]E=0.99E$$

Scattering on light nucleus (^{1}H) – large energy loss (good moderator) – Water used as moderator because it contains H.

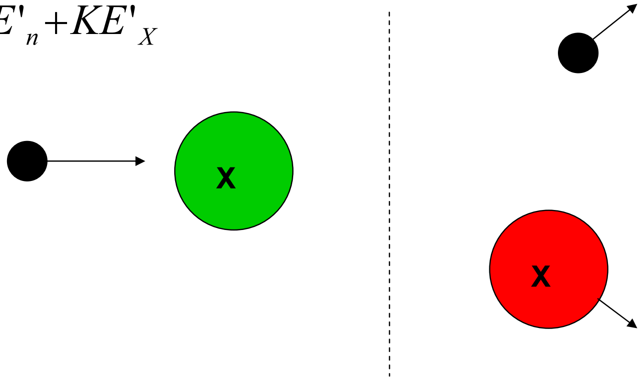
$$\overline{E}'=\frac{1}{2}\left[1+\left(\frac{0}{2}\right)^2\right]E=0.5E$$

Inelastic scattering

$$n+{}_Z^AX\rightarrow n'+{}_Z^AX$$

Kinetic energy is not conserved any more (total energy is)

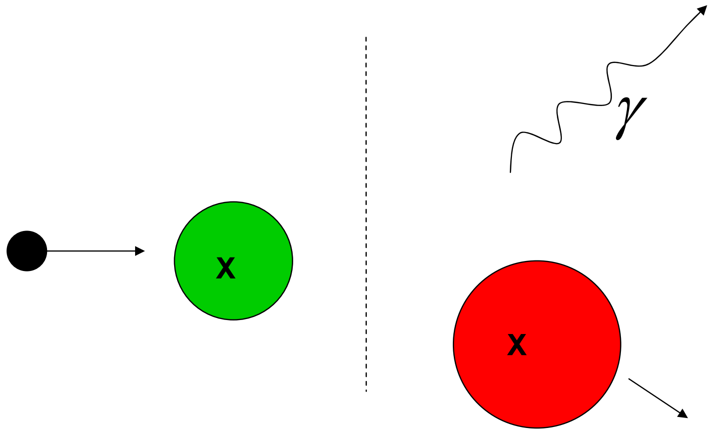
$$KE_n+KE_X>KE'_n+KE'_X$$



The incident neutron is slowed down by inelastic scattering

Radiative Capture

$$n + {}^{A}_{Z}X \rightarrow {}^{A+1}_{Z}X + \gamma$$



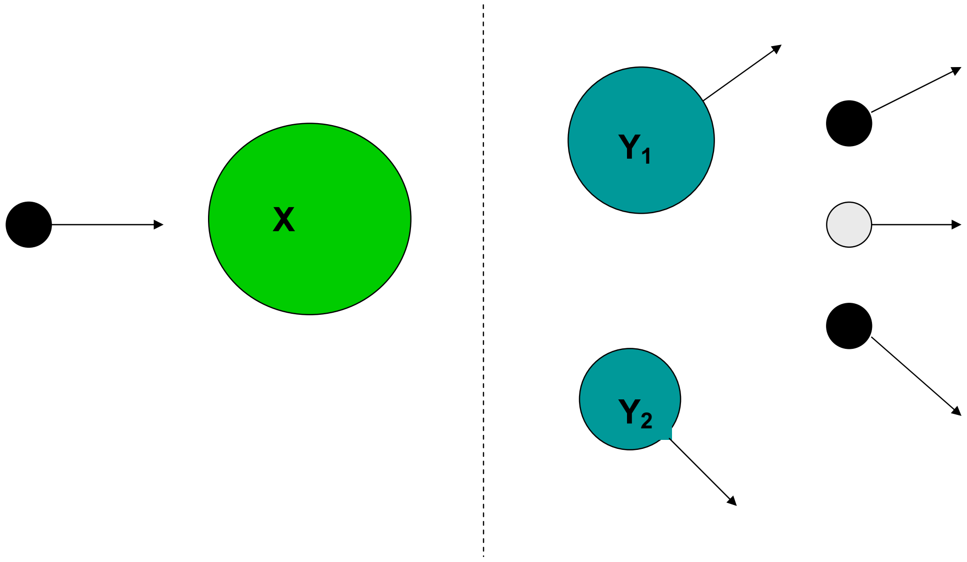
The incident neutron is absorbed (disappears) by radiative capture

Fission

$$n + {}^{A}_{Z}X \rightarrow {}^{A_1}_{Z_1}Y_1 + {}^{A_2}_{Z_2}Y_2 + \nu n + \beta + \gamma + neutrinos + approx.\ 200\,MeV$$

$\nu =$ average number of neutrons ≈ 2.5

(2 to 5 neutrons can be produced)



Fission - Example

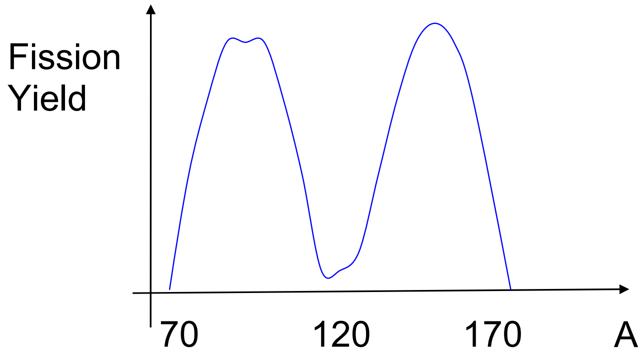
$$ {}_{0}^{1}n + {}_{92}^{235}U \rightarrow X + Y + neutrons $$

- Possible fission reactions

$$ {}_{0}^{1}n + {}_{92}^{235}U \rightarrow {}_{54}^{140}Xe + {}_{38}^{94}Sr + 2n $$

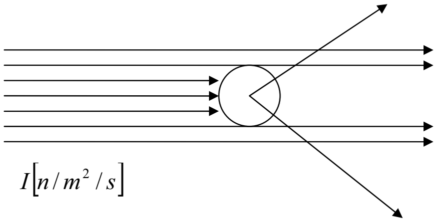
$$ {}_{0}^{1}n + {}_{92}^{235}U \rightarrow {}_{50}^{132}Sn + {}_{42}^{101}Mo + 3n $$

- Distribution of fragments



Microscopic Cross Sections and Reaction Rates for Neutrons

Consider a single nucleus in a parallel beam of monoenergetic neutrons

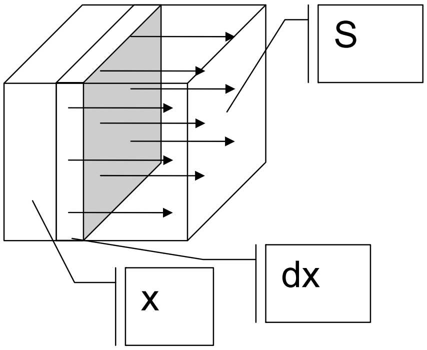


Assume (for now) that scattering and absorption are the only possible reactions.

Neutron Beam Intensity

- Let n(x) be the neutron density (neutrons/cm^3)
- Consider monoenergetic neutrons (All have the same speed)
- Let v be the speed of neutrons.
- Consider a thin "slice" of beam of thickness dx, that crosses surface S.
- There are $dN_n = nSdx$ neutrons in this slice.
- It takes the neutrons in the slice time $dt = \frac{dx}{\mathrm{v}}$ to cross surface S.
- The beam intensity is therefore:

$$I = \frac{dN_n}{Sdt} = \frac{nSdx}{S\frac{dx}{\mathrm{v}}} = n\mathrm{v}$$



Reaction Rates

$$F = R_t = R_s + R_a$$

Probability of a Certain Reaction Type

$$P_s = \frac{R_s}{R_t}$$

$$P_a = \frac{R_a}{R_t}$$

$$P_a + P_s = 1$$

$$P_a + P_s = \frac{R_s + R_a}{R_t} = 1$$

Microscopic Cross sections for Individual Reactions

$$R_t = I\sigma_t$$

$$R_s = R_t P_s = I\sigma_t P_s = I\sigma_s \Rightarrow \sigma_s \equiv \sigma_t P_s$$

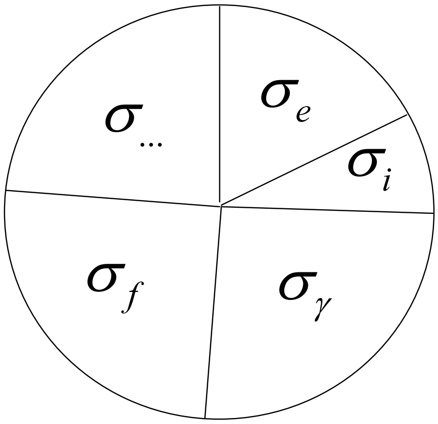
$$R_a = R_t P_a = I\sigma_t P_a = I\sigma_a ; \quad \sigma_a \equiv \sigma_t P_a$$

The sum of individual microscopic cross sections equals the total macroscopic cross section. In our simplified case

$$\sigma_t = \sigma_a + \sigma_s = \sigma_t P_a + \sigma_t P_s = \sigma_t (P_a + P_s) = \sigma_t$$

For the general case:

$$\sigma_t = \sigma_e + \sigma_i + \sigma_\gamma + \sigma_f + \ldots$$



Microscopic Cross Sections as Measures of Probability

We can write:

$$\sigma_t = \frac{R_t}{I}$$

$$\sigma_a = \frac{R_a}{I}$$

$$\sigma_s = \frac{R_s}{I}$$

- The microscopic cross sections can hence be interpreted as the probability of interaction, per unit incident intensity.

Energy Dependence of Microscopic Cross Sections

- The microscopic cross sections depend on the energy of the incident neutrons. ***The nucleus appears larger or smaller depending on how fast the incoming neutron is moving!***

$$\sigma_t = \sigma_t(E)$$

Where E is the kinetic energy of a neutron

$$\sigma_a = \sigma_a(E)$$

$$\sigma_s = \sigma_s(E)$$

Reaction rate per nucleus

$$R(E) = I\sigma(E)$$

The reaction rate depends on the energy (speed) of the incident neutrons.

Volumetric Reaction Rate for a Material
(Collision Density)

Consider a small piece of material placed in a beam of monoenergetic neutrons.

$$F = \frac{R_{\text{single-nucleus}} \times N_{nuclei}}{V} = R_{\text{single-nucleus}} \times N$$

Where N is the number density of nuclei.

$$F = I\sigma \times N \ = I \times \Sigma$$

Where $\Sigma = \sigma \times N$ is the Macroscopic Cross Section.

We have thus recovered the formula obtained in the previous lecture using the attenuation of a collimated beam.

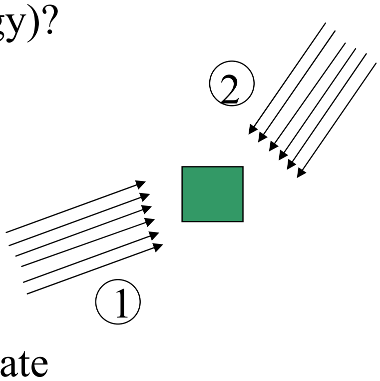
Volumetric Reaction Rate for a Material
(Collision Density)

Dependence on the energy of the incident neutrons

$$F(E) = I \times \sigma(E) \times N \ = I \times \Sigma(E)$$

Neutron flux

Q: What happens if we have a small piece of material bombarded by two beams of monoenergetic neutrons (both having the same energy)?



Reaction (collision) Rate

$$F = I_1\sigma \times N \ + I_2\sigma \times N \ = (I_1 + I_2)\Sigma =$$

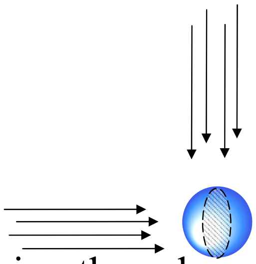
$$= (n_1\text{v} + n_2\text{v})\Sigma = n\text{v}\Sigma = \Phi\Sigma$$

Neutron flux for monoenergetic neutrons: $\Phi = n\text{v}$

Alternative Interpretation of the Neutron Flux for Monoenergetic Neutrons

Consider a small sphere at the intersection of two beams of same-energy monoenergetic neutrons. The situation is similar to having one nucleus bombarded by two neutron beams.

Cross sectional area of sphere: πr^2



The number of neutrons crossing the sphere per second equals the "reaction rate" for the sphere, due to both beams (which is the sum of the reaction rates due to each beam).

Alternative Interpretation of the Neutron Flux for Monoenergetic Neutrons

$$R_{cross} = R_1 + R_2 = I_1 \times \pi r^2 + I_2 \times \pi r^2 = \left(I_1 + I_2\right) \times \pi r^2 =$$

$$= \left(n_1 \mathrm{v} + n_2 \mathrm{v}\right) \times \pi r^2 = n\mathrm{v} \times \pi r^2 = \Phi \times \pi r^2$$

It follows that:

$$\Phi = \frac{R_{cross}}{\pi r^2}$$

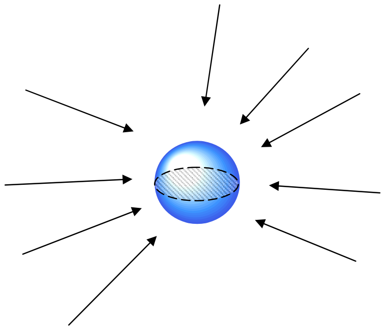
So the flux can also be interpreted as the number of neutrons that cross a sphere per unit time, divided by the cross sectional area of the sphere $\left(\pi r^2\right)$.

Neutron flux for monoenergetic neutrons

For the situation of more than two beams, all of the same energy, the definition of the flux is the same:

$$\Phi = n\text{v}$$

where n is the total neutron density due to all the beams.
The flux can still be interpreted as the number of neutrons crossing a small sphere, divided by the cross section area of the sphere.



Macroscopic Cross Sections for Mixtures

Consider a mixture of nuclei with number densities Ni.
The volumetric reaction rate density for each nucleus type i is:

$$F_i = \Phi\sigma_i \times N_i = \Phi\Sigma_i$$

The total reaction rate density is:

$$F = \sum_i F_i = \sum_i \Phi\sigma_i \times N_i = \Phi\sum_i \sigma_i \times N_i =$$

$$= \Phi\sum_i \Sigma_i = \Phi\Sigma$$

Macroscopic Cross Sections for Mixtures

The total macroscopic cross section equals the sum of the (partial) macroscopic cross sections for each nucleus species

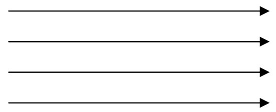
$$\Sigma = \sum_i \Sigma_i = \sum_i N_i \sigma_i$$

$$F = \Phi\Sigma$$

Neutron Intensity, Flux, Current and their Applications

Single Beam

Consider a beam of monoenergetic neutrons



The intensity is given by:

$$I = n\mathrm{v}$$

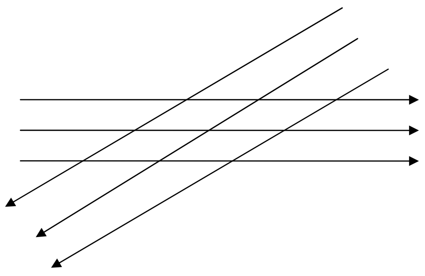
The flux is a scalar quantity given by

$$\Phi = n\mathrm{v}$$

The current is a vectorial quantity given by:

$$\vec{J} = n\vec{\mathrm{v}}$$

Two intersecting beams of different-energy neutrons



Neutron Flux

$$\Phi = n_1\mathrm{v}_1 + n_2\mathrm{v}_2 = \Phi_1 + \Phi_2$$

Neutron Current

$$\vec{J} = n_1\vec{\mathrm{v}}_1 + n_2\vec{\mathrm{v}}_2 = \vec{J}_1 + \vec{J}_2$$

For many intersecting beams:

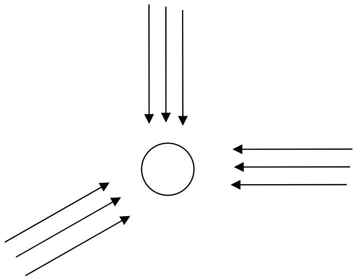
$$\Phi = \sum_i n_i \mathrm{v}_\mathrm{i} = \sum_i \Phi_i$$

$$\vec{J} = \sum_i n_i \vec{\mathrm{v}}_\mathrm{i} = \sum_i \vec{J}_i$$

Usefulness of Neutron Flux

Consider a small sample of material placed at the intersection of several beams of neutrons.

The total collision density in the sample is equal to the sum of the collision densities due to the neutrons in each beam.



$$F = \sum_i F_i$$

We rewrite the expression for the total collision density

$$F = \sum_i F_i = \sum_i \Sigma \times \Phi_i =$$

$$= \Sigma \sum_i \Phi_i = \Sigma\Phi$$

Where $\Phi = \sum_i \Phi_i$

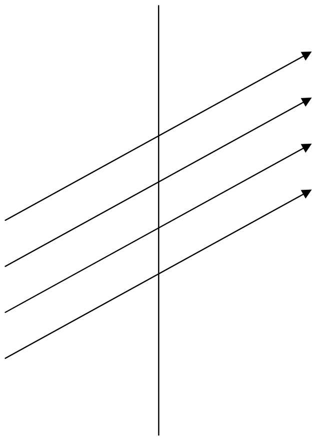
So:

$$F = \Sigma\Phi$$

Regardless of how many beams we have (one or more).

Usefulness of Neutron Current

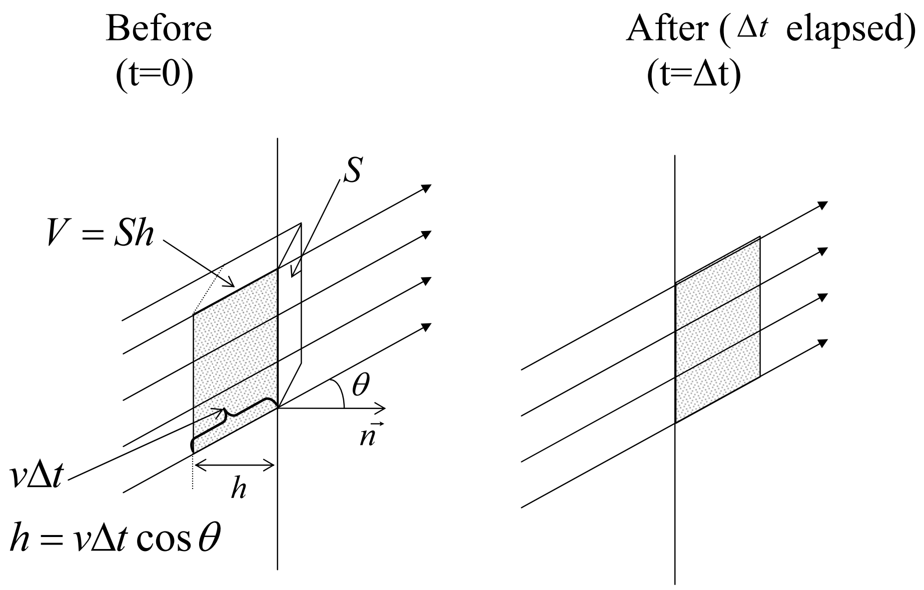
Consider a monoenergetic neutron beam that intersects a plane surface.



We want to determine the rate at which neutrons cross this surface. Per unit area.

$$R = \frac{\Delta N}{S\Delta t}$$

Where ΔN is the number of neutrons crossing the plate at time Δt through surface area *S*.



$\vec{n}$ is the unit vector normal to *S*. $\vec{h} = h\vec{n}$

$$V = Sv\Delta t \cos\theta$$

The rate at which neutrons cross the surface in Δt is given by the neutrons in the marked region.

$$R = \frac{\Delta N}{S\Delta t} = \frac{n\Delta V}{S\Delta t} = \frac{nS\mathrm{v}\Delta t\cos\theta}{S\Delta t} =$$

$$= n\mathrm{v}\cos\theta = n\vec{\mathrm{v}}\vec{n} = (n\vec{\mathrm{v}})\vec{n} = \vec{J}\vec{n}$$

Multiple Beams

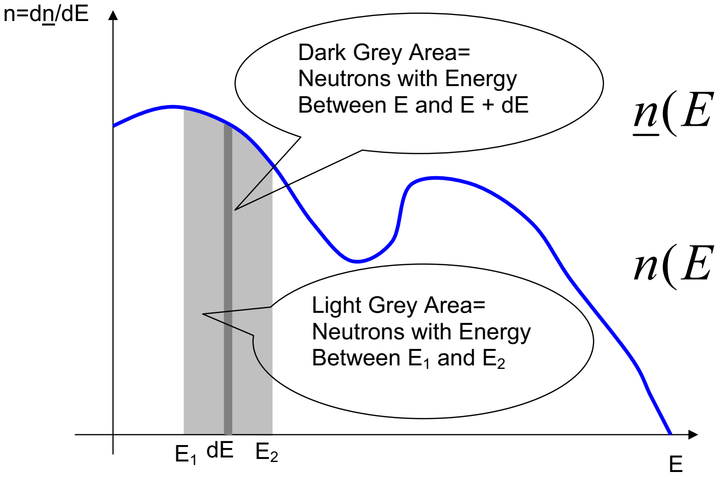
The number of neutrons crossing the surface per unit time per unit area is the sum of the neutrons in each beam that cross the surface per unit time per unit area.

$$R = \sum_i R_i = \sum_i \vec{J}_i\vec{n} =$$

$$= \left(\sum_i \vec{J}_i\right)\vec{n} = \vec{J}\vec{n}$$

Polyenergetic Neutrons

Consider now a parallel beam that has neutrons of different energies (speeds).



$$\underline{n}(E) = \int_0^E n(E')dE'$$

$$n(E) = \frac{d\underline{n}(E)}{dE}$$

(volumetric) density of neutrons with energy less or equal to E.

$$\underline{n}(E)$$

(volumetric) density of neutrons with energy between E and E+dE.

$$d\underline{n} = n(E)dE$$

Neutron density spectrum

$$n(E) = \frac{d\underline{n}(E)}{dE}$$

n is the number of neutrons with energy between *E* and *E+dE,* per unit volume per unit energy (divided by *dE* and by volume).

Beam intensity for neutrons with energy between E and E+dE

$$d\underline{I}(E) = d\underline{n}(E) \times \mathrm{v(E)} = n(E)\mathrm{v}(E)dE$$

The above is the same as eq. 3.36 in the textbook but the textbook does not use the underline.

Energy-dependent beam intensity (Beam Intensity Spectrum)

$$I(E) = \frac{d\underline{I}(E)}{dE} = n(E)\mathrm{v}(E)$$

Total beam intensity:

$$I = \int_0^\infty I(E)dE = \int_0^\infty n(E)\mathrm{v}(E)dE$$

Flux made up of neutrons with energy between E and E+dE:

$$d\Phi(E) = n(E)\mathrm{v}(E)dE$$

Energy-dependent Flux (Flux spectrum)

$$\Phi(E) = n(E)\mathrm{v}(E)$$

Total Flux:

$$\Phi = \int_0^\infty \Phi(E)dE = \int_0^\infty n(E)\mathrm{v}(E)dE$$

Current made up of neutrons with energy between E and E+dE

$$d\vec{J}(E) = n(E)\vec{\mathrm{v}}(E)dE$$

Energy Dependent Current (Current Spectrum)

$$\vec{J}(E) = n(E)\vec{\mathrm{v}}(E)$$

Total current:

$$\vec{J} = \int_0^\infty \vec{J}(E)dE = \int_0^\infty n(E)\vec{\mathrm{v}}(E)dE$$

Reaction rate for neutrons with energy between E and E+dE

$$dR_x(E) = n(E)\mathrm{v}(E)\Sigma_x(E)dE = \Phi(E)\Sigma_x(E)dE$$

Energy-dependent reaction-rate density

$$R_x(E) = n(E)\mathrm{v}(E)\Sigma_x(E) = \Phi(E)\Sigma_x(E)$$

Total Reaction Rate for Reaction x

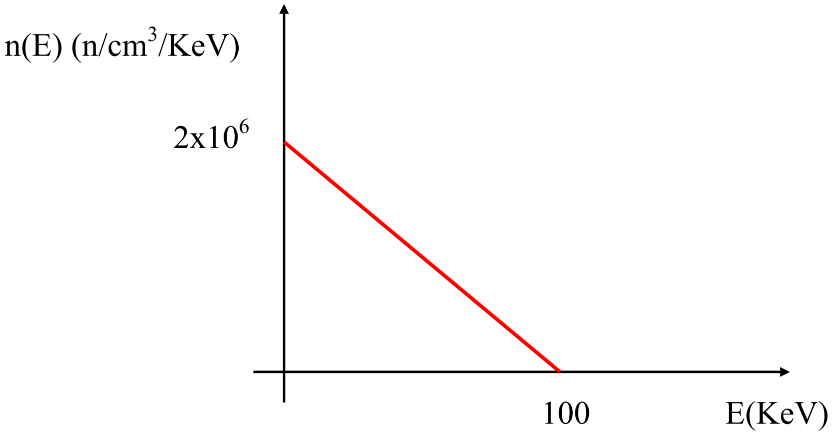
$$R_x = \int n(E)\mathrm{v}(E)\Sigma_x(E)dE = \int_0^\infty \Phi(E)\Sigma_x(E)dE$$

Subscript x can stand for total collisions, or just absorption, or elastic scattering, etc.

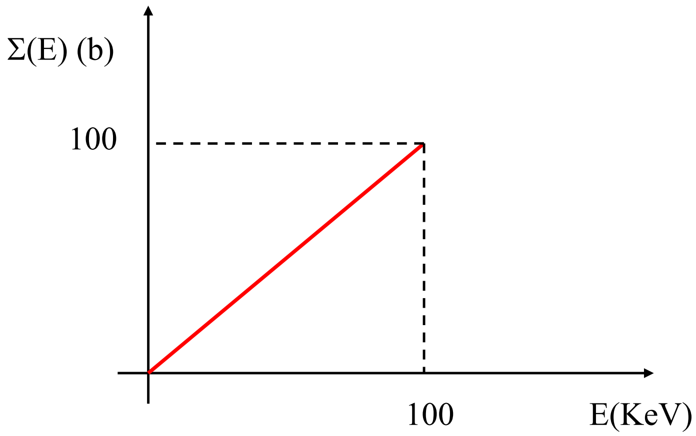
Reaction Rates for Polyenergetic Neutrons: Examples

Example 1

Consider a neutron beam with the following energy dependence of the neutron density:



Assume that the beam is incident on a thin target whose total macroscopic cross section has the following energy dependence:



Calculate:

The total neutron density.
The density of neutrons with energy less than 50 KeV.
The energy dependence of the neutron flux.
The energy dependence of the reaction (collision) rate.
The total collision rate.

The analytical expression for the above is:

$$n(E) = 2 \times 10^{6} \left(1 - \frac{E}{100} \right)$$

Remember:

Volumetric density of neutrons with energy between E and E+dE

$$d\underline{n}(E) = n(E)dE$$

Density of neutrons with energy less than E

$$\underline{n}(E) = \int_{0}^{E} n(E)dE$$

Total neutron density:

$$n_t = \underline{n}(\infty) = \int_{0}^{\infty} n(E)dE$$

Notation abuse (overload):

$$n \equiv n_t = \int_{0}^{\infty} n(E)dE$$

$$n = \int_0^{100} 2\times10^6\left(1-\frac{E}{100}\right)dE + \int_0^{\infty} 0dE$$

$$= 2\times10^6 \int_0^{100}\left(1-\frac{E}{100}\right)dE$$

$$= 2\times10^6\left[E-\frac{E^2}{200}\right]\Bigg|_0^{100}$$

$$= 2\times10^6\times\left[100-\frac{10000}{200}\right]$$

$$= 2\times10^6\times[100-50]$$

$$= 10^8\,\frac{n}{cm^3}$$

$$n < 50 \equiv \underline{n}(50) = \int_0^{50}\cdots = 2\times10^6\left[E-\frac{E^2}{200}\right]\Bigg|_0^{50}$$

$$= 2\times10^6\left[50-\frac{2500}{200}\right]$$

$$= 2\times10^6[50-12.5]$$

$$= 2\times10^6\times37.5$$

$$= 75\times10^6\,\frac{n}{cm^3}$$

Energy Dependence of Φ

$$\Phi(E) = n(E)\mathrm{v}(E)$$

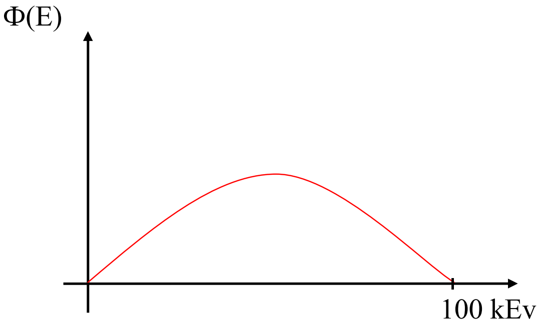
$$E = \frac{m\mathrm{v}^2}{2} \Rightarrow \mathrm{v} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2}{m}}\sqrt{E}$$

$$\Phi(E) = n(E)\sqrt{\frac{2}{m}}\sqrt{E} = \sqrt{\frac{2}{m}}n(E)\sqrt{E}$$

$$= \sqrt{\frac{2}{m}}2\times10^6\left(1-\frac{E}{100}\right)E^{\frac{1}{2}}$$

$$= \sqrt{\frac{2}{m}}2\times10^6\left(E^{\frac{1}{2}}-\frac{E^{\frac{3}{2}}}{100}\right)$$

$$= c\left(E^{\frac{1}{2}}-\frac{E^{\frac{3}{2}}}{100}\right)$$



$$\Phi(100) = c\left(100^{\frac{1}{2}}-\frac{100^{\frac{3}{2}}}{100}\right) = c\left(10-\left(\sqrt{100}\right)^3\frac{1}{100}\right)$$

$$= c\left(10 = \frac{1000}{100}\right) = 0$$

Energy Dependence of Reaction Rate

$$R(E) = \Phi(E)\Sigma(E) = c\left(E^{\frac{1}{2}} - \frac{E^{\frac{3}{2}}}{100}\right)E = c\left(E^{\frac{3}{2}} - \frac{E^{\frac{5}{2}}}{100}\right)$$

$$R(0) = 0$$

$$R(100) = c\left(100^{\frac{3}{2}} - \frac{100^{\frac{5}{2}}}{100}\right) = 0$$

$$R(t) = \int_0^{\infty} R(E)dE$$

Example 2: Problem 34

When thermal neutrons interact with ^{14}N what is the probability that absorption leads to radiative capture?

$$\sigma_t = \sigma_s + \sigma_a = (\sigma_e + \sigma_i) + (\sigma_\gamma + \sigma_f)$$

Captive rate $\Phi\sigma_\gamma$

Total rate $\Phi\sigma_t$

Absorption rate $\Phi\sigma_a$

$$P_{\gamma/a} = \frac{capture\ rate}{absorption\ rate} = \frac{\Phi\sigma_\gamma}{\Phi\sigma_a} = \frac{\sigma_\gamma}{\sigma_a}$$

$$= \frac{\sigma_\gamma}{\sigma_\gamma + \sigma_f} = \frac{1.9}{1.9 + 0} = 1$$

Example 3: Problem 35

Control rods with following $^{w}/_{0}$ composition
5 ($^{w}/_{0}$) Cd
15 ($^{w}/_{0}$) In
80 ($^{w}/_{0}$) Ag
Rate of absorption/g @ 400ºC
Assuming 2200 n/s neutrons (0.025 eV)
$\Phi = 5\times 10^{13} n/cm^2/s$
Ag is a 1/v absorber.

$$R_v = \Phi\Sigma_a$$

$$R_m = \frac{\Phi\Sigma_a}{\rho} = \Phi\frac{N_{Cd}\sigma_a^{Cd} + N_{In}\sigma_a^{In} + N_{Ag}\sigma_a^{Ag}}{N_{Cd}M_{Cd} + N_{In}M_{In} + N_{Ag}M_{Ag}}$$

$$W_i\rho = N_i M_i$$

$$W_{Cd}\rho = N_{Cd}M_{Cd}$$

$$N_{Cd} = \rho\frac{W_{Cd}}{M_{Cd}}$$

$$N_{In} = \rho\frac{W_{In}}{M_{In}}$$

$$N_{Ag} = \rho\frac{W_{Ag}}{M_{Ag}}$$

$$R/g = \Phi \frac{\rho \frac{W_{Cd}}{M_{Cd}} \sigma_{Cd} + \rho \frac{W_{In}}{M_{In}} \sigma_{In} + \rho \frac{W_{Ag}}{M_{Ag}} \sigma_{Ag}}{\rho} =$$

$$= \Phi \left[\frac{W_{Cd}}{M_{Cd}} \sigma_{Cd} + \frac{W_{In}}{M_{In}} \sigma_{In} + \frac{W_{Ag}}{M_{Ag}} \sigma_{Ag} \right] =$$

$$= 5 \times 10^{13} \left[\frac{0.05}{48 amu} 2450 b + \frac{0.15}{49 amu} 183 b + \frac{0.8}{47 amu} 63 b \right] =$$

$$= 5 \times 10^{13} \times 6.022 \times 10^{-24} \left[\frac{0.05}{48} 2450 + \frac{0.15}{49} 183 + \frac{0.8}{47} 63 \right] =$$

$$= 125 \times 10^{12} \frac{abs.}{g}$$

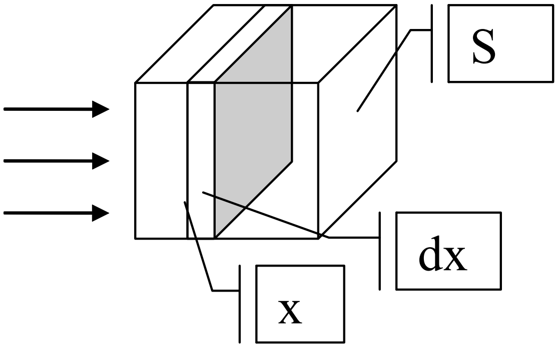
Attenuation of a Neutron Beam from a Neutron Balance Perspective

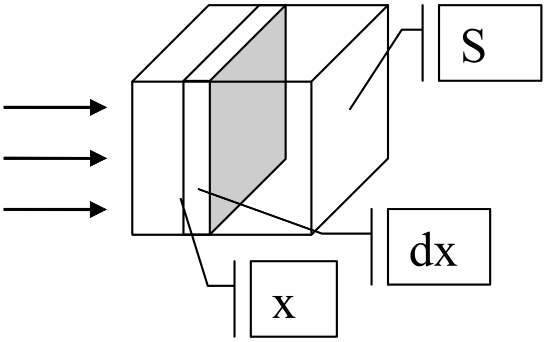
Neutron Attenuation Revisited

Parallel beam of monoenergetic neutrons

For such a beam

$$I \equiv \Phi \equiv J$$





Neutron balance in the volume of thickness dx

$$\underbrace{J(x)S}_{\text{neutrons entering the volume}} - \underbrace{J(x+dx)S}_{\text{neutrons exiting the volume}} = \underbrace{\Sigma(x)\Phi(x)Sdx}_{\text{neutrons colliding (reacting) in the volume}}$$

The neutron balance equation can be rewritten:

$$I(x)S - I(x+dx)S = \Sigma(x)I(x)Sdx$$

Dividing by Sdx on both sides we obtain

$$\frac{I(x) - I(x+dx)}{dx} = \Sigma(x)I(x)$$

Equivalent to:

$$-\frac{dI(x)}{dx} = \Sigma(x)I(x) \Leftrightarrow \frac{dI(x)}{dx} = -\Sigma(x)I(x)$$

If the macroscopic cross section is constant, then:

$$\frac{dI(x)}{dx} = -\Sigma I(x)$$

Which can be integrated to obtain:

$$I(x) = I(0)e^{-\Sigma x}$$

Exactly what we obtained before using a different kind of reasoning.

Moral: If assumptions are right and reasoning correct, the results are the same regardless of the method used.

Mean Free Path

Neutrons that react (collide) between x and x+dx have had a "free path" of length x.

To find the mean free path, we need to average over all the neutrons that interact from x=0 to x=∞.

$$\lambda = \frac{\int_0^\infty x I(x)\Sigma dx}{\int_0^\infty I(x)\Sigma dx} = \frac{\int_0^\infty x I(0)e^{-\Sigma x}\Sigma dx}{\int_0^\infty I(0)e^{-\Sigma x}\Sigma dx} =$$

$$\frac{I(0)\Sigma\int_0^\infty x e^{-\Sigma x}dx}{I(0)\Sigma\int_0^\infty e^{-\Sigma x}dx} = \frac{\int_0^\infty x e^{-\Sigma x}dx}{\int_0^\infty e^{-\Sigma x}dx}$$

The numerator is integrated by parts to give

$$\int_0^\infty x e^{-\Sigma x} dx = \int_0^\infty x \left(-\frac{e^{-\Sigma x}}{\Sigma} \right)' dx =$$

$$= x \left(-\frac{e^{-\Sigma x}}{\Sigma} \right) \Bigg|_0^\infty - \int_0^\infty \left(-\frac{e^{-\Sigma x}}{\Sigma} \right) dx =$$

$$= \left(-\frac{e^{-\Sigma x}}{\Sigma^2} \right) \Bigg|_0^\infty = \frac{1}{\Sigma^2}$$

(Since $\lim_{x \to \infty} (x e^{-\alpha x}) = 0$)

The denominator integrates as:

$$\int_0^\infty e^{-\Sigma x} dx = \left(-\frac{e^{-\Sigma x}}{\Sigma} \right) \Bigg|_0^\infty = \frac{1}{\Sigma}$$

It follows that:

$$\lambda = \frac{\frac{1}{\Sigma^2}}{\frac{1}{\Sigma}} = \frac{1}{\Sigma}$$

Fission

Fission

$${}_{0}^{1}n+{}_{Z_X}^{A_X}X\rightarrow{}_{Z_A}^{A_A}A+{}_{Z_B}^{A_B}B+\tilde{\nu}\,{}_{0}^{1}n+\tilde{\mu}\,{}_{-1}^{0}e+\gamma$$

$\tilde{\nu} = 2, 3, 4, 5$

on average $\nu \cong 2.5$

A & B = Fission Products (Fission Fragments)

Conservation Laws

Number of nucleons

$$A_X + A_n = A_A + A_B + \widetilde{\nu}$$

$$A_X + 1 = A_A + A_B + \widetilde{\nu}$$

Charge

$$Z_X = Z_A + Z_B - \widetilde{\mu}$$

Energy

$$c^2\left[M(n) + M(X)\right] = c^2\left[M(A) + M(B)\right] +$$

$$+ \widetilde{\nu} c^2 M(n) + c^2 \widetilde{\mu} M(e) + E_\gamma$$

M is the relativistic mass

Using the rest mass and kinetic energy E, we have:

$$\begin{aligned}
&c^2\left[M_0(n)+M_0(X)\right]+E_n= \\
&=c^2\left[M_0(A)+M_0(B)\right]+ \\
&+\tilde{\nu}c^2M_0(n)+c^2\tilde{\mu}M_0(e)+ \\
&+E_\gamma+E_A+E_B+E_{\tilde{\nu}n}+E_{\tilde{\mu}e}
\end{aligned}$$

M_0 is the rest mass

The above can be rewritten using the Q:

$$\begin{aligned}
&c^2\left[M_0(n)+M_0(X)\right]+E_n= \\
&=c^2\left[M_0(A)+M_0(B)\right]+ \\
&+\tilde{\nu}c^2M_0(n)+c^2\tilde{\mu}M_0(e)+Q+E_n
\end{aligned}$$

For fission, Q is approximately 200 MeV

Distribution of Energy From Fission

Carrier	**Energy (MeV)**
Fission Fragments	168
Beta	8
Gamma	14
neutrinos	12
neutrons	5
Total	**207**

Most energy is taken by fission fragments and deposited locally.

Fission Mechanism (simplified)

In reality, fission occurs through a compound nucleus which, in turn, can decay very rapidly in several different ways.

$$ {}_{0}^{1}n + {}_{Z_X}^{A_X}X \rightarrow {}_{Z_X}^{A_X+1}X' $$

$$ \overset{\text{(fast)}}{{}_{Z_X}^{A_X+1}X'} \begin{cases} \rightarrow \; A' + B' + \gamma & \text{(mode 1 - prompt } \gamma\text{)} \\ \rightarrow \; A' + B' + \tilde{\nu}_p n & \text{(mode 2 - prompt n)} \end{cases} $$

$$ \nu_p = 2 - 3 $$

Both A' and B' can be stable or further decay in several possible modes:

$$A'\begin{cases} \xrightarrow{\lambda} A & (A'\text{ was stable}) \\ \xrightarrow{\lambda} A+\gamma & (\text{delayed}\,\gamma) \\ \xrightarrow{\lambda} A+{}^{0}_{-1}\beta_{HE} & (\text{mode 3}) \\ \xrightarrow{\lambda} A''+{}^{0}_{-1}\beta_{LE} & (\text{mode 4}) \\ \qquad\downarrow (\text{fast}) & \\ \quad A+n \quad (\text{delayed neutron}) & \end{cases}$$

If A' decays according to mode 4, it is called a *precursor*. There are six possible types of precursor, and six possible values for λ

A'' is then called an *emitter*.

We cannot predict in advance which nuclei will be precursors, but we can predict, on the average how many will do so. This number is equal to the number of delayed neutrons emitted, called the *delayed neutron yield*.

$$\nu_d = \frac{\#\ of\ \ delayed\ \ neutrons}{\#\ of\ \ fissions}$$

We cannot predict how many prompt neutrons will be emitted in each reaction either. But we can predict how many will be produced on the average. This is called the prompt neutron yield.

$$\nu_p = \frac{\#\ of\ \ prompt\ \ neutrons}{\#\ of\ \ fissions}$$

On the average, the fission reaction can be written:

$$n + X \rightarrow A + B + \nu_p n_p + \nu_d n_d + \mu e + \gamma$$

The *total neutron yield* is defined as:

$$\nu = \nu_d + \nu_p \quad \cong 2.5$$

The delayed neutron fraction is:

$$\beta = \frac{\nu_d}{\nu}$$

Delayed Neutrons

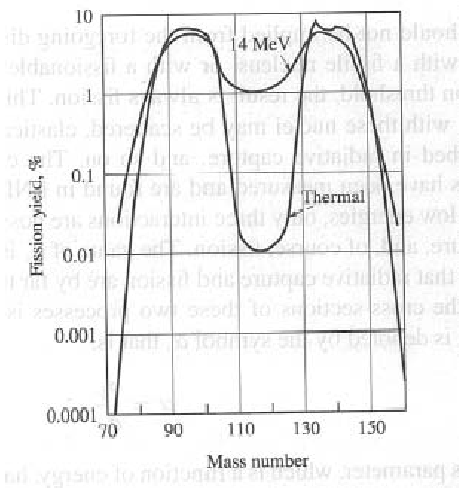
Are emitted by emitters which result from the beta decay of precursors.

There are 6 precursor (delayed neutron) groups, based on their half-life.

$$\nu_d = \nu_{d1} + \nu_{d2} + \nu_{d3} + \nu_{d4} + \nu_{d5} + \nu_{d6}$$

Fission Products (Heavy Nuclei)

Mass is distributed asymmetrically.



^{235}U is *fissile*, i.e. undergoes fission with near-zero energy neutrons with high probability.

$$\sigma_{f \quad \textit{lower energies}} \propto E^{-\frac{1}{2}} \propto \frac{1}{\mathrm{v}}$$

^{238}U is *fissionable*, but not fissile, i.e. it can undergo fission, but with higher energy neutrons and with low probability.

Energy Spectrum of Fission Neutrons

Energy Spectrum

$$\chi(E) \equiv \frac{n(E)}{n_t} = \frac{n(E)}{\int_0^\infty n(E)dE}$$

It follows that:

$$\int_0^\infty \chi(E)dE = \int_0^\infty \frac{n(E)}{\int_0^\infty n(E)dE} dE = \frac{1}{\int_0^\infty n(E)dE} \int_0^\infty n(E)dE = 1$$

Energy Spectrum of Fission Neutrons

Prompt-neutron spectrum (E_{avg}=2MeV)

Important Facts

- Fission neutron energies are much higher than thermal energies (0.025 eV), so they are not appropriate for efficient fission in fissile materials.
- To achieve fission efficiently, the neutrons need to be slowed down (their energy needs to be reduced). This process is called *moderation*. It is achieved by elastic collision with light nuclei (usually Hydrogen or Deuterium)
- Reactors that use thermal neutrons for fission are called *Thermal Reactors*.
- Special reactor designs can be conceived, where fast neutrons are used for fission. These are called *Fast Reactors*.

Fission-Related Parameters

Capture-to-fission ratio

$$\alpha = \frac{\sigma_\gamma}{\sigma_f}$$

Number of neutrons released per absorbed neutron.

$$\eta = \nu \frac{\sigma_f}{\sigma_a}$$

For mixtures of fissile **and** non-fissile elements:

$$\eta = \frac{1}{\Sigma_a} \sum_i \nu_i \Sigma_{fi}$$

Nuclear Reactors – The Basics

Nuclear Reactors

- Can be of two Types:
  - **Thermal** - fissions induced by thermal ($E<1eV$) neutrons in fissile nuclei
  - **Fast** - fissions induced by fast ($E\cong 1MeV$) in fissile/fissionable nuclei

Thermal Reactor Components

- ***Fuel*** - consists of nuclei that fission liberating energy
- ***Moderator*** - slows down fast neutrons resulting from fission to thermal energies so they can fission fuel nuclei
- ***Coolant*** - removes the heat

The three can be:
- mixed together → Homogeneous Reactor
- separated → Heterogeneous Reactor

Most reactors are **heterogeneous**.

Power Reactors

- **Pressurized Water Reactors**
- ***Pressurized Heavy-Water Reactors (CANDU)***
- Gas-Cooled Reactors
- Other

CANDU Reactors

- Heterogeneous

- *Fuel:* Natural Uranium Oxide
  - (UO_2 0.7% ^{235}U, 99.3% ^{238}U)
- *Coolant:* Heavy Water (D_2O)
- *Moderator:* Heavy Water (D_2O)

CANDU Reactor - How it Works

- Fissions take place in the fuel
- Most energy from fissions is taken up by fission fragments which stop in less that one micron.
- In stopping, the fission fragments' kinetic energy becomes heat, which raises the fuel temperature.
- The fuel is cooled by the coolant, which takes the heat from the fuel to the steam generators.
- Neutrons are also produced from fission.
- Fission neutrons are slowed-down by elastic collisions in the moderator and, to a smaller extent, in coolant.
- Once they become thermal, neutrons can induce new fissions, keeping the chain reaction going.

CANDU Reactor - How it Works (cont.)

- Part of the neutrons get absorbed by radiative capture or "leak" out of the reactor. These do not induce fissions.
- On the average, only one neutron per each fission succeeds in inducing a new fission, so there is a uniform rate of fissions and not an avalanche of fissions.

Neutron Diffusion and Moderation

Nomenclature

General Nomenclature

Consider a quantity, say the number of collisions N_{coll}:

Rate

We call *rate*, the ratio between the amount of that quantity that is found or produced between time t and time t+dt and dt. (i.e. the collision rate is the ratio between the number of collisions that occur between t and t+dt divided by dt):

$$R_{coll} = \frac{dN_{coll}}{dt}$$

Spectrum

We call (energy) *spectrum* the ratio between the amount of that quantity that is found or produced between energy E and E+dE and dE (i.e. the collision spectrum is the ratio between the number of collisions suffered by neutrons with energies between E and E+de and dE):

$$F(E) = \frac{dN_{coll}}{dE}$$

Density

We call (volumetric) *density*, the ratio between the total quantity dQ existing or produced in volume dV and dV (i.e. the collision density is the ratio between the number of collisions suffered by neutrons in volume dV and dV)

$$F = \frac{dN_{coll}}{dV}$$

We can have names that imply double ratios, e.g.

Collision *density spectrum*. - the ratio between the number of collisions suffered by neutrons in dV with energies between E and E+dE and dVdE

$$F(E) = \frac{dN_{coll}}{dVdE}$$

Collision *density rate*:

$$F = \frac{dN_{coll}}{dVdt}$$

Oftentimes, when talking about double ratios people omit to name one of them, so you must pay attention to the context.

For example, one will often refer to the collision rate or collision density, when, in fact, meaning collision density rate.

The same letter is sometimes used to denote different quantities.

Always look at the context.

Recapitulation of Important Concepts

Recapitulation of Important Concepts

Volumetric total reaction (collision) rate density for monoenergetic neutrons

$$F = \Sigma_t \Phi$$

or

$$F = \Sigma_t n \mathrm{v}$$

Reaction rate for neutrons with energies between E and E+dE:

$$dF = \Sigma(E) \times n(E)dE \times \mathrm{v(E)}$$

(Total) Reaction rate for neutrons of all energies:

$$F = \int_0^\infty \Sigma_t(E)n(E)\mathrm{v(E)dE} = \int_0^\infty \Sigma_\mathrm{t}(E)\Phi(E)dE$$

where

$$\Phi(E) = n(E)\mathrm{v(E)}$$

Reaction Rates for Individual Reactions

Scattering reaction rate density:

$$F_s = \int_0^\infty \Sigma_s(E)\Phi(E)dE$$

Absorption reaction rate density (number of neutrons absorbed per cm^3 per s):

$$F_a = \int_0^\infty \Sigma_a(E)\Phi(E)dE$$

Neutron Diffusion

Fick’s Law

Fick's Law (Diffusion Law)

- Will accept it without proof.
- Valid far from interfaces.
- Valid for materials with relatively low absorption.

Gives the neutron current as a function of the neutron flux

Assume monoenergetic neutrons
Assume the flux only varies along the x axis:

$$J_x = -D\frac{d\Phi(x)}{dx}$$

D = Diffusion Coefficient

In three dimensions (and monoenergetic neutrons):

$$\vec{J} = -D grad\Phi = -D\nabla\Phi$$

Definition of gradient:

$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f}{dx} \\ \frac{\partial f}{dy} \\ \frac{\partial f}{dz} \end{bmatrix}$$

Number of particles crossing a surface of orientation $\vec{n}$ per unit time per unit area (normal current):

$$J_n = \vec{J} \cdot \vec{n}$$

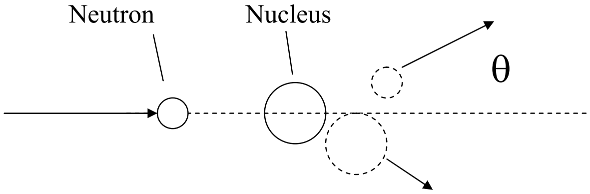
$$D = \frac{\lambda_{tr}}{3}, \quad (\lambda = \frac{1}{\Sigma})$$

Transport mean free path

$$\lambda_{tr} = \frac{1}{\Sigma_{tr}} = \frac{1}{\Sigma_s (1 - \overline{\mu})}$$

Average of the cosine of the scattering angle

$$\overline{\mu} = \overline{\cos\theta}$$



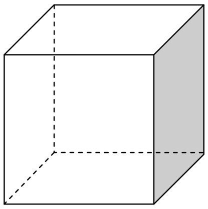
$$\overline{\mu} = \frac{2}{3A}$$

Diffusion Equation

Neutron Balance Equation (equation of Continuity) for Monoenergetic Neutrons

Expresses the conservation of neutrons

$$
\begin{aligned}
&\left[\text{Rate of change in the number of neutrons in a small volume dV}\right] = \\
&= \left[\text{Rate of neutron production in volume dV}\right] - \\
&- \left[\text{Rate of neutron absorption in volume dV}\right] - \\
&\left[\text{Rate of neutron leakage from dV}\right]
\end{aligned}
$$

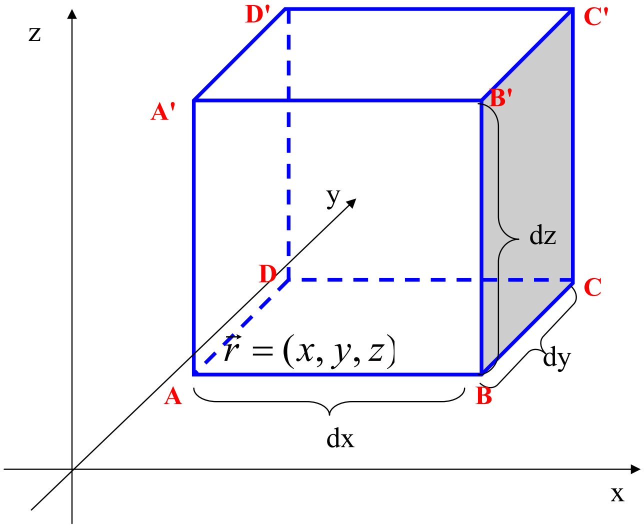


$$dV = dxdydz$$

Will follow a derivation slightly different from the one in the textbook.

You are welcome to use the derivation in the book. Brush up on your vector calculus and Gauss' formula if you want to follow the derivation in the book.

Infinitesimal (i.e. very small) Volume



$$dV = dxdydz$$

dx, dy, dz small-enough that:

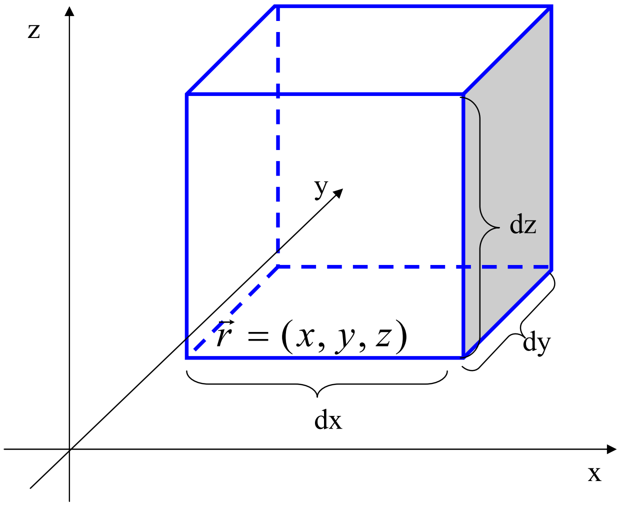
$$\Phi(x, y, z) \cong \Phi(x + dx, y, z)$$

$$\vec{J}(x, y, z) \cong \vec{J}(x + dx, y, z)$$

and similarly for y and z

Production

The number of neutrons being produced per unit time in volume dV. (Neutron Source)



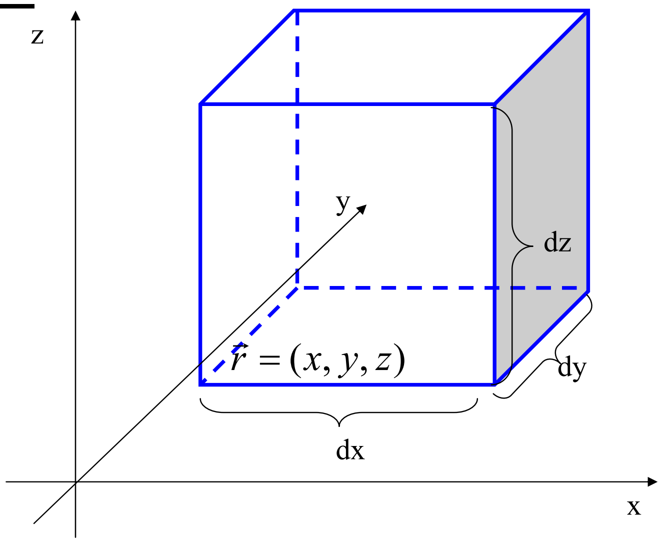
Production Rate:

SdV

$s(cm^{-3}s^{-1})$

Production Rate $= SdV = Sdxdydz$

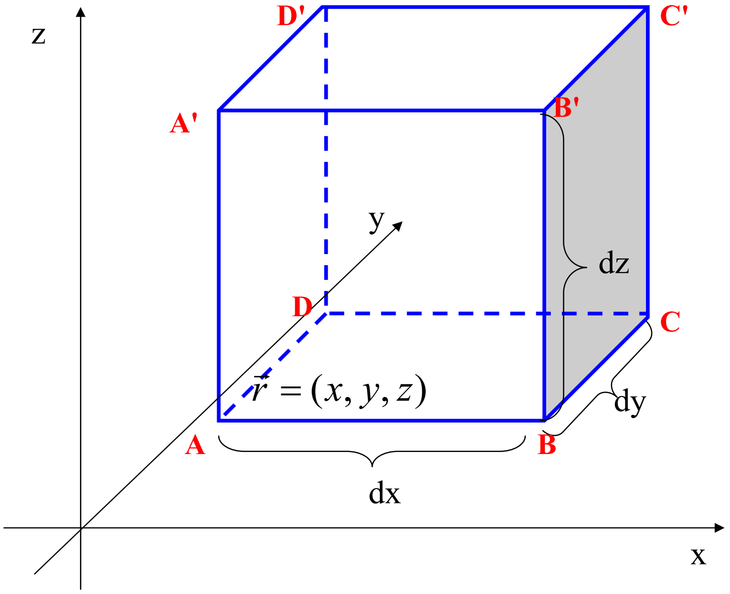
Absorption



Assume, dx, dy, dz are small enough that the flux Φ varies negligibly inside our volume

$$R_a = \Sigma_a \Phi dV = \Sigma_a \Phi dxdydz$$

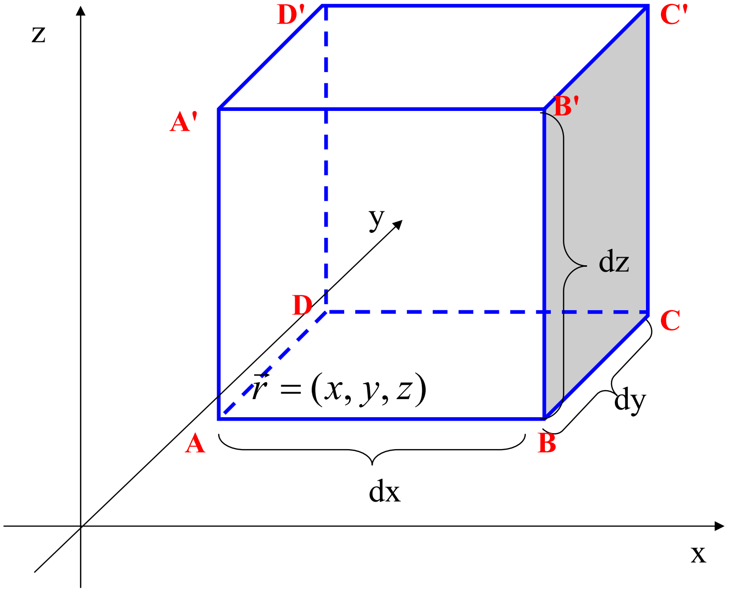
Leakage Through Face BCC'B'



$$LK_{x+} = LK_{BCC'B'} \cong \vec{J}\left(x+dx, y+\frac{dy}{2}, z+\frac{dz}{2}\right)\cdot \hat{u}_x dydz$$

$$LK_{x+} = LK_{BCC'B'} \cong J_x\left(x+dx, y, z\right)dydz$$

Leakage Through Face ADD'A'



$$LK_{x-} = LK_{ADD'A'} \cong \vec{J}\left(x, y+\frac{dy}{2}, z+\frac{dz}{2}\right)\cdot\left(-\hat{u}_x\right)dydz$$

$$LK_{x-} = LK_{ADD'A'} \cong -J_x\left(x, y, z\right)dydz$$

Net Leakage Along X Axis

$$LK_x = LK_{x+} + LK_{x-} = LK_{BCC'B'} + LK_{ADD'A'} \cong$$
$$\cong J_x\left(x+dx,y,z\right) - J_x\left(x,y,z\right)dydz$$

Let's remember that:

$$\frac{\partial J_x}{\partial x}\left(x,y,z\right) = \frac{J_x\left(x+dx,y,z\right) - J_x\left(x,y,z\right)dydz}{dx}$$

Hence:

$$J_x\left(x+dx,y,z\right) - J_x\left(x,y,z\right)dydz = \frac{\partial J_x}{\partial x}(x,y,z)dx$$

$$LK_x \cong J_x\left(x+dx,y,z\right) - J_x\left(x,y,z\right)dydz =$$
$$= \frac{\partial J_x}{\partial x}(x,y,z)dxdydz$$

Total Leakage out of dV

$$LK = LK_x + LK_y + LK_z =$$
$$= \frac{\partial J_x}{\partial x}(x,y,z)dxdydz + \frac{\partial J_y}{\partial y}(x,y,z)dxdydz + \frac{\partial J_z}{\partial z}(x,y,z)dxdydz =$$
$$= \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right)\Bigg|_{x,y,z} dxdydz = \left(div\vec{J}\right)dxdydz = \left(\nabla\cdot\vec{J}\right)dxdydz$$

Definition of divergence for a vector function $\vec{f}(x,y,z)$:

$$div\vec{f} \equiv \nabla \cdot \vec{f} = \left(\frac{\partial f_x}{dx} + \frac{\partial f_y}{dy} + \frac{\partial f_z}{dz} \right)$$

Rate of Change of Number of Neutrons in dV

$$R_{change} = \frac{\#neutrons(t+dt) - \#neutrons(t)}{dt} = \frac{n(t+dt)dV - n(t)dV}{dt} =$$

$$= \frac{n(t+dt) - n(t)}{dt} dV = \frac{\partial n}{\partial t} dxdydz$$

Neutron Balance Equation for dV

$$\frac{\partial n}{\partial t} dxdydz = sdxdydz - \Sigma_a \Phi dxdydz - \nabla \cdot \vec{J} dxdydz$$

Dividing by the volume $dV = dxdydz$ we obtain:

$$\frac{\partial n}{\partial t} = s - \Sigma_a \Phi - \nabla \cdot \vec{J}$$

Valid regardless of whether Fick's law holds true or not

Neutron Balance in the Diffusion Approximation

Assume Fick's Law to be true:

$$\vec{J} = -D \nabla \Phi$$

Substitute into the neutron balance eq:

$$\frac{\partial n}{\partial t} = -\nabla \cdot \left(-D \nabla \Phi \right) - \Sigma_a \Phi + s$$

This is the *time-dependent diffusion equation for monoenergetic neutrons.*

It is important because by solving it we find the flux and the flux allows us to calculate all reaction rates, including fission rate - which is really what we are after, by using $R = \Sigma \Phi$.

If the diffusion coefficient is constant:

$$\frac{\partial n}{\partial t} = D\nabla \cdot \left(\nabla \Phi\right) - \Sigma_a \Phi + s$$

Remember the definition of the Laplacian:

$$\Delta f(x,y,z) \equiv \nabla^2 f = \nabla \cdot \left(\nabla f\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The diffusion eq. can then be rewritten:

$$\frac{\partial n}{\partial t} = D\nabla^2 \Phi - \Sigma_a \Phi + s$$

If we keep in mind that

$$\phi = n\mathrm{v} \Rightarrow n = \frac{\Phi}{\mathrm{v}} \Rightarrow \frac{\partial n}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\Phi}{V}\right) = \frac{1}{V}\frac{d\Phi}{dt}$$

We obtain:

$$\frac{1}{\mathrm{v}}\frac{\partial \phi}{\partial t} = D\nabla^2 \Phi - \Sigma_a \Phi + s$$

Steady-State Situation (no time dependence)

$$D\nabla^2 \Phi - \Sigma_a \Phi + s = 0$$

Steady-State diffusion equation for monoenergetic neutrons and constant D

Dividing by D:

$$\nabla^2\Phi - \frac{\Sigma_a}{D}\Phi + \frac{s}{D} = 0$$

Introducing notation (Diffusion Length):

$$L^2 = \frac{D}{\Sigma_a}$$

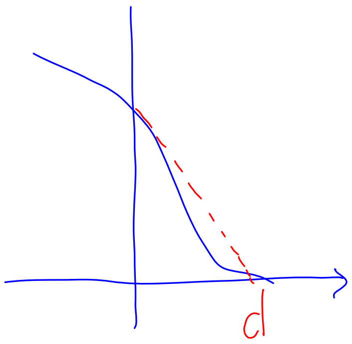
$$-\nabla^2\Phi + \frac{1}{L^2}\Phi = \frac{s}{D}$$

Interface Conditions for the Diffusion equation:

Continuity of flux: $\Phi_A = \Phi_B$

Continuity of normal component of current $J_{A\perp} = J_{B\perp}$

Vacuum Interface



$$\Phi(d) = 0$$

Extrapolation distance $d = 0.71\lambda_{tr}$

$$d = 2.13D$$

$$\lambda_{tr} = \frac{1}{\Sigma_{tr}}; \quad \Sigma_{tr} = \Sigma_s(1-\bar{\mu}); \quad D = \frac{1}{3\Sigma_{tr}} \Rightarrow \Sigma_{tr}\frac{1}{3D}$$

The Concept of Infinite Homogeneous Medium

Medium is the same at any point
Hence, there is no reason why the flux would be different an any particular point

$$\Phi(x,y,z)=\Phi=const$$

The current is given by Fick's Law

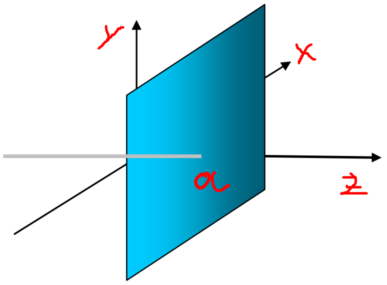
$$\vec{J}=\nabla\Phi=\begin{bmatrix}\dfrac{\partial\Phi}{\partial x}\\ \dfrac{\partial\Phi}{\partial y}\\ \dfrac{\partial\Phi}{\partial z}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=0$$

The current is zero in an infinite homogeneous medium

$$\Sigma_a\Phi=S;\quad \nabla^2\Phi=0;\quad \Phi=\frac{S}{\Sigma_a}$$

The Concept of Homogeneous Half Space

$$x\in(-\infty,\infty)$$
$$y\in(-\infty,\infty)$$
$$z\in(a,\infty)$$



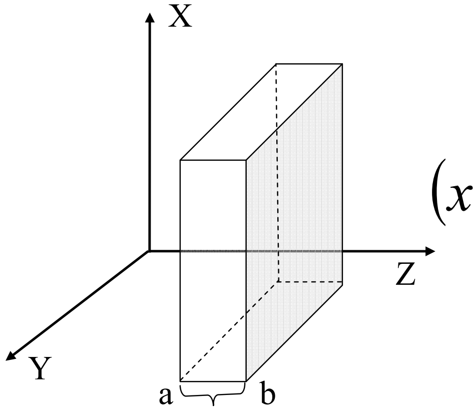
In such a configuration, since for the same z all points are identical, there is no variation in the flux with x or y

$$\Phi(x,y,z)=\Phi(z)\,;\qquad \nabla^2\Phi=\left(0+0+\frac{\partial^2\Phi}{\partial z^2}\right)$$

$$D\frac{\partial^2\Phi(z)}{\partial z^2}-\Sigma_a\Phi(z)+S(z)=0$$

The Concept of Infinite Homogeneous Slab

- finite in Z, but infinite in X and Y directions



$$(x,y,z)\left|\begin{array}{l} x\in(-\infty;\infty)\\ y\in(-\infty;\infty)\\ z\in(a;b)\end{array}\right.$$

Because there is no change in the material properties in either X or Y direction,

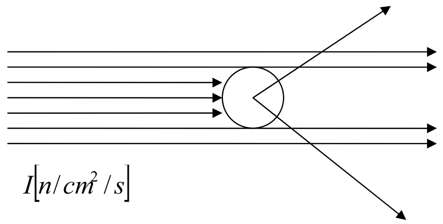
$$\Phi(x,y,z)=\Phi(z)$$

Energy-Dependent Diffusion

Differential Microscopic Scattering Cross Sections

Beam of monoenergetic neutrons

$$I = n\mathrm{v}(E)$$



Scattering rate:

$$R_s = I\sigma_s(E)$$

Equivalently, we can write (using only macroscopic quantities that can be measured):

$$\sigma_s(E) = \frac{R_s}{I}$$

By scattering, neutrons lose energy.

Let $dR_s(E')$ be the rate at which neutrons are scattered in energy range E', E'+dE'

We have:

$$\int_0^\infty dR_s(E') = R_s$$

Definition of the differential scattering microscopic cross section

$$\sigma_s(E \to E') \equiv \frac{dR_s(E')}{IdE'}$$

Equivalently, we can write:

$$\sigma_s(E \to E') \equiv \frac{dR_s(E')}{IdE'}\frac{R_s}{R_s} = \frac{R_s}{I}\frac{dR_s(E')}{R_s dE'} = \sigma_s(E)\frac{dR_s(E')}{R_s dE'}$$

Scattering Kernel

$$k(E \to E') = \frac{dR_s(E')}{R_s dE'} = \frac{dP(E, E')}{dE'}$$

$$\sigma_s(E \to E') = \sigma_s k(E \to E')$$

The scattering kernel can be interpreted as the probability density function for a neutron of energy E to be scattered such that its final energy is between E' and E'+dE'.

The differential and total scattering cross section satisfy:

$$\sigma_s(E) = \int_0^\infty \sigma_s(E \to E') dE'$$

Differential Macroscopic Scattering Cross Sections

$$\Sigma_s(E \to E') = N\sigma_s(E \to E')$$

or, using the scattering kernel:

$$\Sigma_s(E \to E') = N\sigma_s(E)k(E \to E') =$$
$$= \Sigma_s(E)k(E \to E')$$

Volumetric reaction rate at which neutrons scatter within energy range (E, E+dE)

$$R_s(E \to E') = I\Sigma_s(E \to E')$$

Energy-Dependent Neutron Balance Equation

Balance Equation for Neutrons with Energy Between E and E+dE

[rate of change of number of neutrons in volume dV with energy within range (E, E+dE)] =
[rate of production in volume dV of neutrons with energy within range (E, E+dE)] +
[rate of scattering of neutrons in dV into energy range (E, E+dE)] -
[rate of absorption in dV of neutrons with energy in range (E, E+dE)] -
-[rate of scattering of neutrons in dV outside of energy range (E, E+dE)] -
[rate of leakage out of dV of neutrons with energy within range (E, E+dE)]

[rate of change of number of neutrons in volume dV with energy within range (E, E+dE)]

$$R_{change}(E) = \frac{n(E,t+dt)dEdV - n(E,t)dEdV}{dt} =$$

$$= \frac{n(E,t+dt) - n(E,t)}{dt} dV = \frac{\partial n(E)}{\partial t} dEdV$$

[rate of production in volume dV of neutrons with energy within range (E, E+dE)]

$$R_p(E) = s(E)dEdV$$

s(E) = number of neutrons produced inside dV with energies between E and E+dE, divided by dEdV.

[rate of scattering of neutrons in dV into energy range (E, E+dE)]

Rate at which neutrons with energy within $(E';E'+dE)$ scatter such that their energy is within $(E;E+dE)$

$$R_s(E'\to E)=\Phi(E')dE'\times\Sigma_s(E'\to E)dEdV$$

Rate at which all neutrons scatter such that their energy is within $(E;E+dE)$

$$R_s(\to E)=\left[\int_0^\infty \Phi(E')\times\Sigma_s(E'\to E)dE'\right]dEdV$$

[rate of absorption in dV of neutrons with energy in range (E, E+dE)]

$$R_a(E)=\Phi(E)dE\times\Sigma_a(E)dV=\Phi(E)\times\Sigma_a(E)dEdV$$

[rate of scattering of neutrons in dV outside of energy range (E, E+dE)]

$$R_s(E \rightarrow) = \Phi(E)dE \times \Sigma_s(E)dV = \Phi(E) \times \Sigma_s(E)dEdV$$

Note that:

$$\Sigma_s(E) = \int_0^\infty \Sigma_s(E \rightarrow E')dE'$$

[rate of leakage out of dV of neutrons with energy within range (E, E+dE)]

$$LK(E) = \nabla \cdot \vec{J}(E)dEdV$$

Balance Equation for Neutrons with Energy Between E and E+dE

$$R_{change}(E) = R_p(E) + R_s(\rightarrow E) - R_a(E) - R_s(E \rightarrow) - LK(E)$$

$$\begin{aligned}\frac{\partial n(E)}{\partial t} dEdV &= s(E)dEdV + \left[\int_0^\infty \Phi(E') \times \Sigma_s(E' \rightarrow E)dE'\right] dEdV - \\ &- \Phi(E) \times \Sigma_a(E)dEdV - \Phi(E) \times \Sigma_s(E)dEdV - \nabla \cdot \vec{J}(E)dEdV\end{aligned}$$

Dividing by dEdV we obtain the energy-dependent neutron balance equation (continuity equation):

$$\begin{aligned}\frac{\partial n(E)}{\partial t} &= s(E) + \int_0^\infty \Phi(E') \times \Sigma_s(E' \rightarrow E)dE' - \\ &- \Phi(E) \times \Sigma_a(E) - \Phi(E) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E)\end{aligned}$$

We can show the dependence on time explicitly:

$$\begin{aligned}\frac{\partial n(E,t)}{\partial t} &= s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \rightarrow E)dE' - \\ &- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E,t)\end{aligned}$$

Definition of energy-dependent flux:

$$\Phi(E) = n(E)\mathrm{v(E)} \Rightarrow n(E) = \frac{\Phi(E)}{\mathrm{v(E)}}$$

Substituting the expression for the energy-dependent neutron density, we obtain:

$$\begin{aligned}\frac{1}{\mathrm{v}(E)}\frac{\partial \Phi(E,t)}{\partial t} &= s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \rightarrow E)dE' - \\ &- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E,t)\end{aligned}$$

Energy-Dependent Steady-State Neutron Balance Equation

$$0 = s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \to E) dE' -$$
$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) - \nabla \cdot \vec{J}(E,t)$$

Diffusion Approximation (use Fick's Law)

$$\vec{J}(E) = D(E) \nabla \Phi(E)$$

$$0 = s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \to E) dE' -$$
$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) + \nabla \cdot \left(D(E) \nabla \Phi(E) \right)$$

For position-independent diffusion coefficient:

$$0 = s(E,t) + \int_0^\infty \Phi(E',t) \times \Sigma_s(E' \to E) dE' -$$
$$- \Phi(E,t) \times \Sigma_a(E) - \Phi(E,t) \times \Sigma_s(E) + D(E) \nabla^2 \Phi(E)$$

Multigroup Formalism

Approximate treatment of the energy-dependent diffusion equation.

Energy Groups

Divide the energy domain $(0, E_{\max})$ into intervals called *groups*

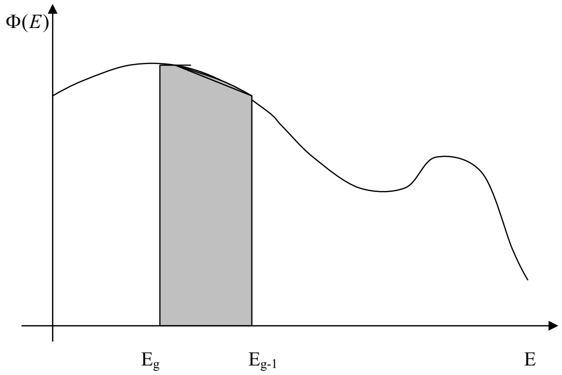
$$0 = E_G < E_{G-1} \ldots\ldots < E_2 < E_1 < E_0 = E_{\max}$$

Group neutron density

$$n_g \equiv \int_{E_g}^{E_{g-1}} n(E)\,dE$$

(Energy) Group Flux

$$\Phi_g \equiv \int_{E_g}^{E_{g-1}} \Phi(E)dE$$



Can depend on parameters such as position and/or time

$$\Phi_g(\vec{r}) \equiv \int_{E_g}^{E_{g-1}} \Phi(\vec{r}, E)dE$$

Group Current

$$\vec{J}_g \equiv \int_{E_g}^{E_{g-1}} \vec{J}(E)dE$$

Can depend on parameters such as position and/or time

$$\vec{J}_g(\vec{r}) \equiv \int_{E_g}^{E_{g-1}} \vec{J}(\vec{r}, E)dE$$

Group Source

$$s_g = \int_{E_g}^{E_{g-1}} s(E)dE$$

Group Reaction Rates

Reaction Rate for a single Nucleus

$$R_g^{\text{single nucleus}} \equiv \int_{E_g}^{E_{g-1}} R(E)dE = \int_{E_g}^{E_{g-1}} \Phi(E)\sigma(E)dE$$

Reaction Rate Density for a Material

$$R_g \equiv \int_{E_g}^{E_{g-1}} R(E)dE = \int_{E_g}^{E_{g-1}} \Phi(E)\Sigma(E)dE$$

Can depend on parameters such as position and/or time

$$R_g(\vec{r}) \equiv \int_{E_g}^{E_{g-1}} R(\vec{r},E)dE = \int_{E_g}^{E_{g-1}} \Phi(\vec{r},E)\Sigma(\vec{r},E)dE$$

Group Cross Sections

Microscopic Group Cross Sections

$$\sigma_g \equiv \frac{\int_{E_g}^{E_{g-1}} \Phi(E)\sigma(E)dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE} = \frac{R_g^{\text{single nucleus}}}{\Phi_g}$$

Macroscopic Group Cross Sections

$$\Sigma_g \equiv \frac{\int_{E_g}^{E_{g-1}} \Phi(E)\Sigma(E)dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE} = \frac{R_g}{\Phi_g}$$

Inter-Group Scattering (Transfer) Cross Sections

Microscopic

$$\sigma_{g\to g'} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_{g'}}^{E_{g'-1}} \Phi(E)\sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE}$$

Macroscopic

$$\Sigma_{g\to g'} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_{g'}}^{E_{g'-1}} \Phi(E)\Sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE} \qquad g \neq g'$$

Intra-Group Scattering Cross Section

Microscopic

$$\sigma_{g\to g} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_g}^{E_{g-1}} \Phi(E)\sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE}$$

Macroscopic

$$\Sigma_{g\to g} = \frac{\int_{E_g}^{E_{g-1}} \left[\int_{E_g}^{E_{g-1}} \Phi(E)\Sigma_s(E \to E')dE' \right] dE}{\int_{E_g}^{E_{g-1}} \Phi(E)dE}$$

Multigroup Neutron balance Equation

[rate of change of number of neutrons in volume dV with energy within group g] =
[rate of production in volume dV of neutrons with energy within group g] +
[rate of scattering of neutrons in dV into energy group g] -
[rate of absorption in dV of neutrons with energy in group g] -
-[rate of scattering of neutrons in dV outside of energy group g] -
[rate of leakage out of dV of neutrons with energy within group g]

Multigroup Neutron balance Equation

$$\frac{\partial}{\partial t} n_g dV = s_g dV + \sum_{g'=1}^{G} \Sigma_{sg'\to g} \Phi_{g'} dV - \Sigma_{ag} \Phi_g dV - \Sigma_{sg} \Phi_g dV - \nabla \cdot \vec{J}_g dV$$

Dividing by dV:

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1}^{G} \Sigma_{sg'\to g} \Phi_{g'} - \Sigma_{ag} \Phi_g - \Sigma_{sg} \Phi_g - \nabla \cdot \vec{J}_g$$

Multigroup Neutron balance Equation

Multigroup Fick's Law:

$$\vec{J}_g = -D_g \nabla \Phi_g$$

Multigroup Diffusion Equation

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1}^{G} \Sigma_{sg' \to g} \Phi_{g'} - \Sigma_{ag} \Phi_g - \Sigma_{sg} \Phi_g + \nabla \cdot \left(D_g \nabla \Phi_g \right)$$

For constant diffusion coefficient:

$$\frac{\partial}{\partial t} n_g = s_g + \sum_{g'=1}^{G} \Sigma_{sg' \to g} \Phi_{g'} - \Sigma_{ag} \Phi_g - \Sigma_{sg} \Phi_g + D_g \nabla^2 \Phi_g$$

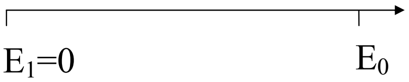
Steady state (no time dependence)

$$-D_g \nabla^2 \Phi_g + \Sigma_{ag} \Phi_g + \Sigma_{sg} \Phi_g - \sum_{g'=1}^{G} \Sigma_{sg' \to g} \Phi_{g'} = s_g$$

Particular Cases of the Diffusion Equation

One-Group Diffusion Equation

The entire energy range is included in just one group



Time-dependent:

$$\frac{\partial n}{\partial t} = S + \Sigma_s \Phi - \Sigma_a \Phi - \Sigma_s \Phi + D\nabla^2 \Phi$$

$$\frac{\partial n}{\partial t} = S - \Sigma_a \Phi + D\nabla^2 \Phi \Rightarrow \mathit{one\ group\ D.E}$$

One-Group Diffusion Equation

Steady State:
The steady-state multigroup diffusion equation

$$-D_g\nabla^2\Phi_g - \sum_{g'=1}^{g-1}\Sigma_{sg'\to g}\Phi_{g'} + \Sigma_{ag}\Phi_g + \sum_{g'=g+1}^{G}\Sigma_{sg\to g'}\Phi_g = s_g$$

becomes:

$$-D_1\nabla^2\Phi_1 + \Sigma_{a1}\Phi_1 = s_1$$

We can drop the group index to obtain:

$$-D\nabla^2\Phi + \Sigma_a\Phi = s$$

Two-group Diffusion Equation

<u>Group 1</u> (fast group):   g=1

$$\frac{\partial n_1}{\partial t} = S_1 + \left(\Sigma_{s_{1\to1}}\Phi_1 + \Sigma_{s_{2\to1}}\Phi_2\right) - \Sigma_{a_1}\Phi_1 - \Sigma_{s_1}\Phi_1 + D_1\nabla^2\Phi_1$$

Where $\Sigma_{s_1} = \Sigma_{s_{1\to2}} + \Sigma_{s_{1\to1}}$

$$\frac{\partial n_1}{\partial t} = S_1 + \Sigma_{s_{1\to1}}\Phi_1 + \Sigma_{s_{2\to1}}\Phi_2 - \Sigma_{a_1}\Phi_1 - \Sigma_{s_{1\to2}}\Phi_1 - \Sigma_{s_{1\to1}}\Phi_1 + D_1\nabla^2\Phi_1$$

$$\frac{\partial n_1}{\partial t} = S_1 - \Sigma_{a_1}\Phi_1 - \Sigma_{s_{1\to2}}\Phi_1 + D_1\nabla^2\Phi_1$$

Group 2 (Slow, thermal)

$$\frac{\partial n_2}{\partial t} = S_2 + \Sigma_{s_{1\to 2}}\Phi_1 + \Sigma_{s_{2\to 2}}\Phi_2 - \Sigma_{a_2}\Phi_2 - \Sigma_{s_2}\Phi_2 + D_2\nabla^2\Phi_2$$

Where $\Sigma_{s_2} = \Sigma_{s_{2\to 1}} + \Sigma_{s_{2\to 2}}$

$$\frac{\partial n_2}{\partial t} = S_2 + \Sigma_{s_{1\to 2}}\Phi_1 + \Sigma_{s_{2\to 2}}\Phi_2 - \Sigma_{a_2}\Phi_2 - \Sigma_{s_{2\to 1}}\Phi_2 - \Sigma_{s_{2\to 2}}\Phi_2 + D_2\nabla^2\Phi$$

$$\frac{\partial n_2}{\partial t} = S_2 + \Sigma_{s_{1\to 2}}\Phi_1 - \Sigma_{a_2}\Phi_2 + D_2\nabla^2\Phi_2$$

Two-group diffusion equation

$$\frac{\partial n_1}{\partial t} = S_1 - \Sigma_{a_1}\Phi_1 - \Sigma_{s_{1\to 2}}\Phi_1 + D_1\nabla^2\Phi_1$$

$$\frac{\partial n_2}{\partial t} = S_2 + \Sigma_{s_{1\to 2}}\Phi_1 - \Sigma_{a_2}\Phi_2 + D_2\nabla^2\Phi_2$$

$\Sigma_{a_1} + \Sigma_{s_{1\to 2}} = \Sigma_r$ (Removal cross section)

Two-group diffusion equation

Steady state $\frac{\partial n}{\partial t} = 0$

$$\begin{cases} -D_1\nabla^2\Phi_1 + \Sigma_{r_1}\Phi_1 = S_1 \\ -D_2\nabla^2\Phi_2 + \Sigma_{a_2}\Phi_2 = S_2 + \Sigma_{s_{1\to 2}}\Phi_1 \end{cases}$$

Two-Group Diffusion Equation

We could have started directly with the steady-state multigroup diffusion equation

$$-D_g\nabla^2\Phi_g - \sum_{g'=1}^{g-1}\Sigma_{sg'\to g}\Phi_{g'} + \Sigma_{ag}\Phi_g + \sum_{g'=g+1}^{G}\Sigma_{sg\to g'}\Phi_g = s_g$$

Group 1 (fast):

$$-D_1\nabla^2\Phi_1 + \Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = s_1$$

Group 2 (slow, thermal):

$$-D_2\nabla^2\Phi_2 - \Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = s_2$$

Two-group diffusion equations:

$$-D_1\nabla^2\Phi_1 + \Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = s_1$$

$$-D_2\nabla^2\Phi_2 - \Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = s_2$$

Solving the Diffusion Equation for Simple Cases

One Group, Infinite Homogeneous Medium, Uniformly Distributed Source

$$-D\nabla^2\Phi(\vec{r})+\Sigma_a\Phi(\vec{r})=s$$

Infinite, homogeneous medium

$$\Phi(\vec{r})=\Phi=ct$$

$$\nabla^2\Phi=0$$

The equation becomes:

$$\Sigma_a\Phi=s$$

Solving for the flux, we obtain:

$$\Phi=\frac{s}{\Sigma_a}\ \text{(constant)}$$

Two Groups, Infinite Homogeneous Medium, Uniformly Distributed Source

$$-D_1\nabla^2\Phi_1+\Sigma_{a1}\Phi_1+\Sigma_{s1\to 2}\Phi_1=s_1$$
$$-D_2\nabla^2\Phi_2-\Sigma_{s1\to 2}\Phi_1+\Sigma_{a2}\Phi_2=s_2$$

For an infinite and homogeneous medium with uniformly-distributed source:

$$\Phi_1(\vec{r})=\Phi_1=ct$$
$$\Phi_2(\vec{r})=\Phi_2=ct$$
$$\nabla^2\Phi_1=0$$
$$\nabla^2\Phi_2=0$$

The two-group equations become:

$$\Sigma_{a1}\Phi_1+\Sigma_{s1\to 2}\Phi_1=s_1$$
$$-\Sigma_{s1\to 2}\Phi_1+\Sigma_{a2}\Phi_2=s_2$$

The first equation can be easily solved to yield:

$$\Phi_1=\frac{s_1}{\Sigma_{a1}+\Sigma_{s1\to 2}}=\frac{s_1}{\Sigma_r}$$

$\Sigma_r =$ removal cross section

The second equation can be rewritten as:

$$\Sigma_{a2}\Phi_2 = s_2 + \Sigma_{s1\to 2}\Phi_1$$

$$\Sigma_{s1\to 2}\Phi_1 = q_T = \text{slowing down density}$$

Using the expression found for the fast flux, we have:

$$\Sigma_{a2}\Phi_2 = s_2 + \Sigma_{s1\to 2}\frac{s_1}{\Sigma_r}$$

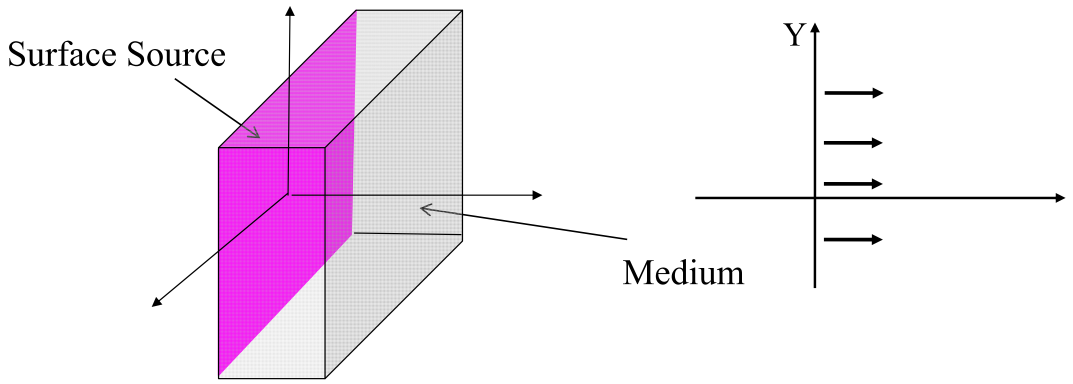
The thermal flux is hence:

$$\Phi_2 = \frac{s_2 + \Sigma_{s1\to 2}\frac{s_1}{\Sigma_r}}{\Sigma_{a2}}$$

If there is no external thermal source $(s_2 = 0)$, then the solution simplifies to:

$$\Phi_2 = \Sigma_{s1\to 2}\frac{s_1}{\Sigma_r\Sigma_{a2}}$$

One-group diffusion equation for a semi-infinite medium (half space)



$$\Phi(x, y, z) = \Phi(z)$$

$$-D\frac{d^2\Phi}{dz^2} + \Sigma_a\Phi = 0$$

→ Assume no volume sources

$$-\frac{d^2\Phi}{dz^2} + \frac{\Sigma_a\Phi}{D} = 0$$

$$L^2 = \frac{D}{\Sigma_a}$$

→ Diffusion Length

$$-\frac{d^2\Phi}{dz^2} + \frac{\Sigma_a\Phi}{L^2} = 0$$

Characteristic Equation

$$-r^2 + \frac{1}{L^2} = 0$$

$$r^2 = \frac{1}{L^2}$$

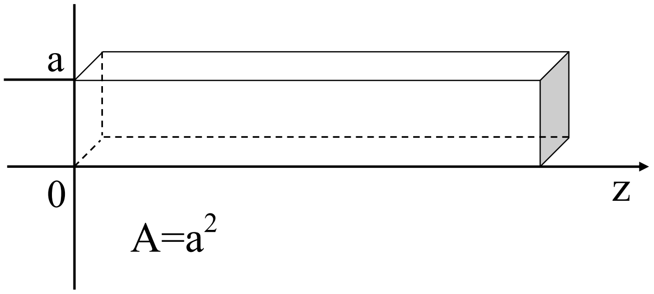
$$r = \pm\frac{1}{L}$$

General solution is

$$ae^{\frac{z}{L}} + be^{-\frac{z}{L}} \Rightarrow be^{-\frac{z}{L}}$$

Finding the constant b:

Consider an infinite parallelipiped of cross section area A and extending from zero to infinity in the z direction :



Express the equality between total absorption in the parallelipiped and the source of neutrons coming in from the boundary source S_b at Z=0.

Absorption rate from 0 to ∞ in a prism of cross-section area A

$$\left[\int_0^\infty \Sigma_a \Phi(z) dz \right] A$$

Source rate

$$S_b A$$

Equality between source and absorption:

$$S_b A = \left[\int_0^\infty \Sigma_a \Phi(z) dz \right] A$$

$$S_b = \int_0^\infty \Sigma_a \Phi(z) dz = \int_0^\infty \Sigma_a b e^{-\frac{z}{L}} dz = \Sigma_a b \int_0^\infty e^{-\frac{z}{L}} dz$$

$$\int_0^\infty e^{-\frac{z}{L}} dz = \left[-Le^{-\frac{z}{L}} \right] \Bigg|_0^\infty = -\lim_{z \to 0} \left(Le^{-\frac{z}{L}} \right) + Le^{-\infty} = L$$

$$S_b = \Sigma_a b L \Rightarrow b = \frac{S_b}{\Sigma_a L}$$

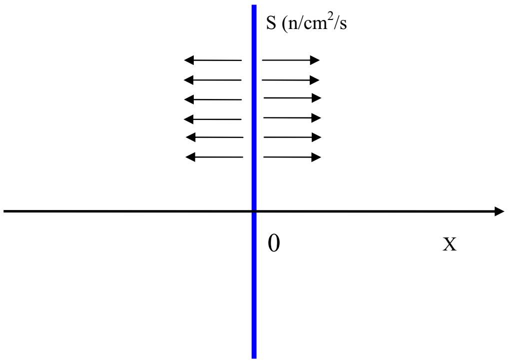
Hence:

$$\Phi(z) = \frac{S_b}{\Sigma_a L} e^{-\frac{z}{L}}$$

Units:

$$[\Phi] = \frac{[S]}{[\Sigma][L]} = \frac{\frac{n}{cm^2 s}}{\frac{1}{cm} cm} = \frac{n}{cm^2 s}$$

One Group Diffusion for an Infinite Planar Source Situated in an Infinite Homogeneous Medium at x=0



Equivalent to two half-spaces (left and right)

$$-D\nabla^2\Phi(x,y,z)+\Sigma_a\Phi(x,y,z)=0 \quad \text{for x} \neq 0$$

Because of the planar (y-z) symmetry, $\Phi=\Phi(x)$

The equation becomes:

$$-D\frac{\partial^2\Phi(x)}{\partial\Phi^2(x)}+\Sigma_a\Phi(x)=0$$

Using the diffusion length notation:

$$\frac{d^2\Phi}{dx^2}-\frac{1}{L^2}\Phi=0, \quad x\neq 0$$

This is a homogeneous second order linear differential equation with constant coefficients. The general solution is of the type:

$$\Phi(x) = Ae^{\frac{-x}{L}} + Ce^{\frac{x}{L}}$$

Because the flux needs to be finite, we have C=0. Hence:

$$\Phi(x) = Ae^{-x/L}$$

The current is:

$$J(x) = -D\frac{d}{dx}\left(Ae^{-x/L}\right) = A\frac{D}{L}e^{-x/L}$$

To find A, we use the boundary condition:

$$\lim_{x\to+0} J(x) = \frac{S}{2}$$

The initial condition yields:

$$J(x) = A\frac{D}{L}e^{0/L} = \frac{s}{2} \Rightarrow A\frac{D}{L} = \frac{s}{2} \Rightarrow A = \frac{sL}{2D}$$

The flux for x>0 is hence:

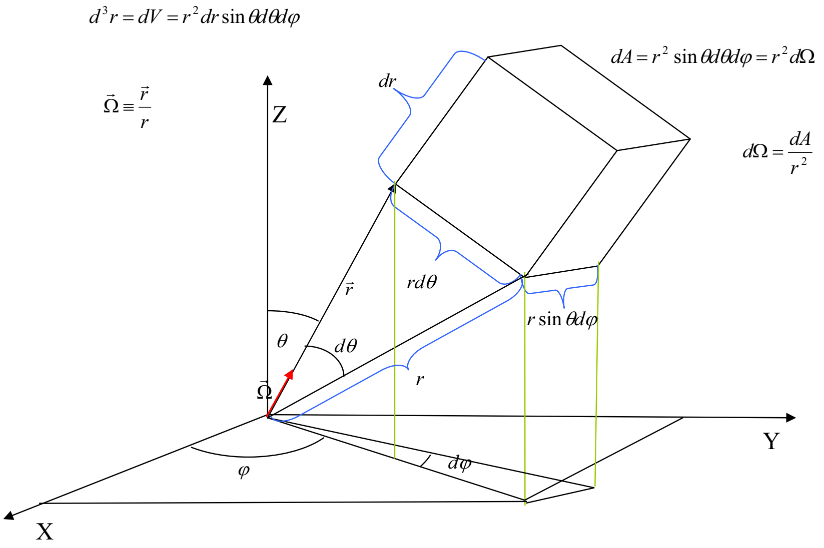
$$\Phi = \frac{SL}{2D}e^{-x/L}$$

Analogously, the flux for x<0 is:

$$\Phi = \frac{SL}{2D}e^{x/L}$$

One Group Diffusion for a Point Source Situated in an Infinite Homogeneous Medium

Use spherical coordinates with the source placed at the center



Because the problem is symmetrical with respect to both θ and φ (spherical symmetry), the flux will only depend on r.

$$\Phi(x, y, z) \Rightarrow \Phi(r, \theta, \varphi)$$

$$\Phi = \Phi(r)$$

Expression of Laplacian in spherical coordinates for a function with spherical symmetry, f(r).

$$\nabla^2 f(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right)$$

The diffusion equation becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) - \frac{1}{L^2}\Phi = 0$$

This is a homogeneous second order differential equation with constant coefficients.

The boundary condition is

$$J 4\pi\varepsilon^2 = S \Rightarrow \varepsilon^2 J = \frac{S}{4\pi}$$

$$\lim_{r\to 0}\left(r^2 J(r)\right) = \frac{S}{4\pi}$$

Rate at which neutrons exit a very small sphere of radius ε, surrounding the origin: $S = J 4\pi\varepsilon^2$

This is equal to the rate at which neutrons are produced, because ε is so small that absorption in this very small sphere can be ignored.

To solve the equation, we make the substitution:

$$w = r\Phi \Leftrightarrow \Phi = \frac{w}{r}$$

The equation becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{w}{r}\right)\right] - \frac{1}{L^2}\frac{w}{r} = 0$$

which yields:

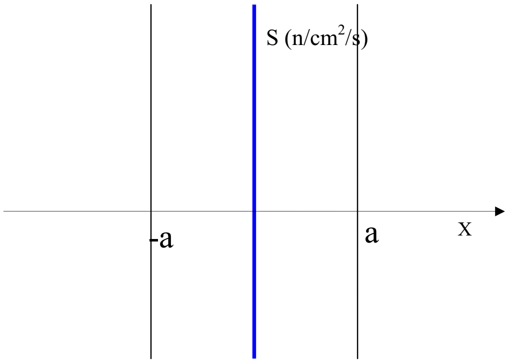
$$\frac{d^2w}{dr^2} - \frac{1}{L^2}w = 0$$

Solution is: $$w = Ae^{-\frac{r}{L}}; \quad \Phi = \frac{Ae^{-\frac{r}{L}}}{r}$$

Following a similar treatment as for the plane source, we find:

$$\Phi = \frac{S}{4\pi D}\frac{e^{-\frac{r}{L}}}{r}$$

One Group Diffusion for a Bare Slab with an Infinite Planar Source Situated in the Middle



The problem is symmetric with respect to the source and also has planar symmetry

$$\Phi = \Phi(x)$$

Diffusion equation:

$$\frac{d^2\Phi}{dx^2} - \frac{1}{L^2}\Phi = 0, \quad x \neq 0$$

Will treat the right half.

This is a homogeneous second order linear differential equation with constant coefficients. The general solution is of the type:

$$\Phi(x) = Ae^{\frac{-x}{L}} + Ce^{\frac{x}{L}}$$

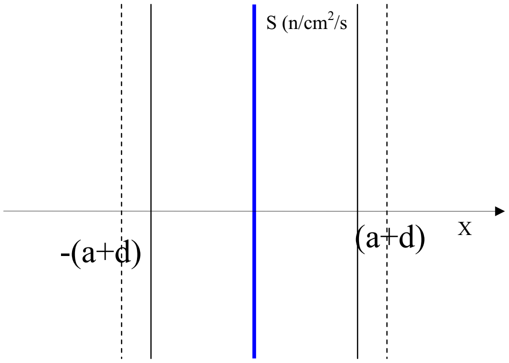
The left boundary condition is, just as before:

$$\lim_{x \to +0} J(x) = \frac{S}{2}$$

yielding:

$$J(x) = -\frac{D}{L} A e^{\frac{-x}{L}} + \frac{D}{L} C e^{\frac{x}{L}} \Bigg|_{x=0} = -\frac{D}{L} A + \frac{D}{L} C = \frac{S}{2}$$

The right boundary condition is now a vacuum boundary condition, that is the flux vanishes at the extrapolated boundary.



$$\phi(\widetilde{a}) = 0$$

where

$$\widetilde{a} = a + d$$

The above yields:

$$\Phi(\tilde{a}) = Ae^{\frac{-\tilde{a}}{L}} + Ce^{\frac{\tilde{a}}{L}} = 0$$

We obtain A and C by solving the system:

$$-\frac{D}{L}A + \frac{D}{L}C = \frac{S}{2}$$

$$\Phi(\tilde{a}) = Ae^{\frac{-\tilde{a}}{L}} + Ce^{\frac{\tilde{a}}{L}} = 0$$

The final solution is:

$$\Phi(x) = \frac{SL}{2D}\frac{e^{\frac{x}{L}} - e^{\frac{x-2\tilde{a}}{L}}}{1 + e^{-\frac{2\tilde{a}}{L}}}$$

Neutron Moderation (two group treatment)

Two-group diffusion

Assume $\Sigma_a = 0$ (good moderator)

$$
\begin{aligned}
-D_1\nabla^2\Phi_1 + \Sigma_{s1\to 2}\Phi_1 &= 0\\
-D_2\nabla^2\Phi_2 - \Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 &= 0
\end{aligned}
$$

The two equations can be rearranged to:

$$
\begin{aligned}
-D_1\nabla^2\Phi_1 + \Sigma_{s1\to 2}\Phi_1 &= 0\\
-D_2\nabla^2\Phi_2 + \Sigma_{a2}\Phi_2 &= \Sigma_{s1\to 2}\Phi_1
\end{aligned}
$$

We make the following notations:

$$\frac{D_1}{\Sigma_{s1\to 2}} = \tau_T \quad = \text{age}$$

$$\frac{D_2}{\Sigma_{a2}} = L_T^2 \quad = \text{thermal diffusion area}$$

$$L_T = \text{thermal diffusion lenght}$$

With the new notations, the equations are written:

$$
\begin{aligned}
-\nabla^2\Phi_1 + \frac{1}{\tau_T}\Phi_1 &= 0\\
-\nabla^2\Phi_2 + \frac{1}{L_T^2}\Phi_2 &= \frac{D_1}{D_2\tau_T}\Phi_1
\end{aligned}
$$

These can be solved for different configurations.

Nuclear Reactor Theory

Multiplication Constant

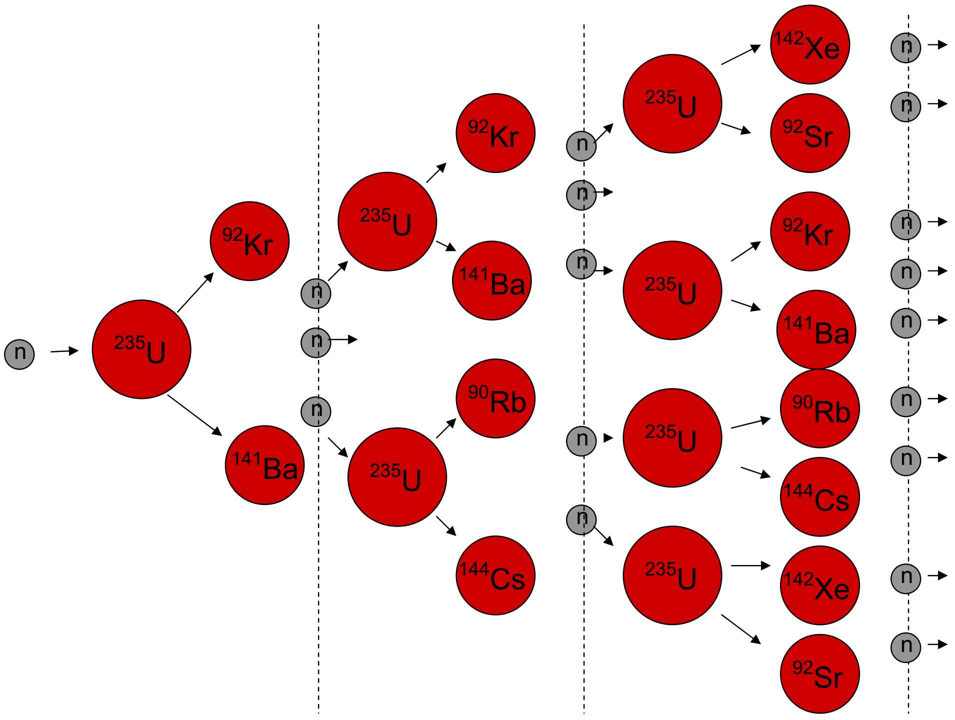
Preliminaries - Neutron Fluence

Neutron *fluence* is defined as the time integral of the flux

$$\psi = \int_{t_1}^{t_2} \Phi(t)dt$$

Where units for ψ are $\frac{n}{cm^2}$ and units for Φ are $\frac{n}{cm^2 s}$

Fission Chain Reaction



Each fission produces 2-3 more neutrons which can, in principle, induce new fissions in avalanche. This is not desirable.

However, not all neutrons resulting from fission induce new fissions. Some undergo gamma capture.

If two few neutrons (less than one per fission) induce new fissions the fission reaction dies down. Not desirable either.

The trick is to only allow one of the secondary neutrons to induce a new fission and thus have a fission rate that is constant in time. A reactor operating at a constant fission rate is said to be *critical*.

Infinite Homogeneous Reactor
(One-Group Diffusion Approximation)

Multiplicative medium ($\Sigma_f > 0$).

Non-Multiplicative medium ($\Sigma_f = 0$).

The steady-state diffusion equation is written:

$$-D\nabla^2\Phi + \Sigma_a\Phi = S$$

The source now consists of fission neutrons:

$$S = \nu\Sigma_f\Phi$$

So the equation becomes:

$$-D\nabla^2\Phi + \Sigma_a\Phi = \nu\Sigma_f\Phi$$

The flux is constant in space because the medium is infinite and homogeneous, so the equation becomes.

$$\Sigma_a \Phi = \nu \Sigma_f \Phi$$

It is obvious that the above cannot be satisfied, unless

$$\Sigma_a = \nu \Sigma_f$$

If that is not the case, then the source is artificially divided by a factor k, just to balance the equation.

$$\Sigma_a \Phi = \frac{1}{k} \nu \Sigma_f \Phi$$

k is called the *multiplication constant (factor)*. For an infinite medium, it is called the *infinite multiplication constant* and denoted by k_∞.

It is obvious that, for the one-group homogeneous reactor case:

$$k_\infty = \frac{\nu \Sigma_f}{\Sigma_a}$$

It is also obvious that the value of the flux cannot be determined because once the appropriate k is used, any value of the flux will satisfy the balance equation.

$$\Sigma_a \Phi = \frac{1}{k_\infty} \nu \Sigma_f \Phi \Rightarrow \Sigma_a \Phi = \frac{1}{\frac{\nu \Sigma_f}{\Sigma_a}} \nu \Sigma_f \Phi$$

Interpretation of k

Since the balance equation is written:

$$\Sigma_a \Phi = \frac{1}{k_\infty} \nu \Sigma_f \Phi$$

We have:

$$k_\infty = \frac{\nu \Sigma_f \Phi}{\Sigma_a \Phi} = \frac{production\,rate}{loss\ rate}$$

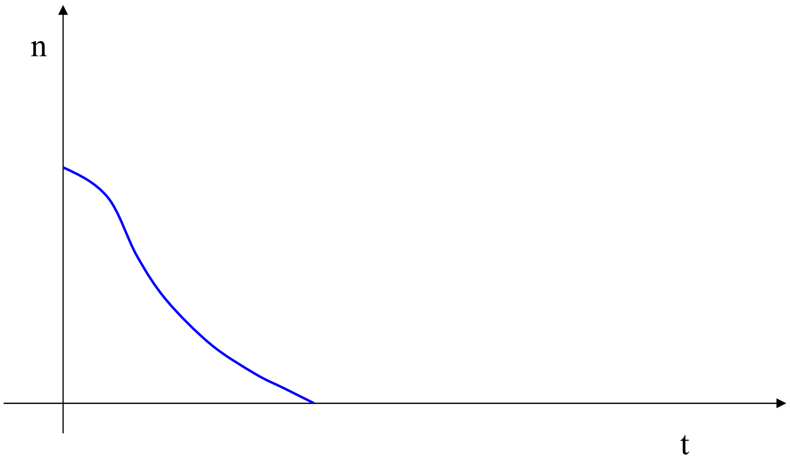
So k can be interpreted as the ratio of the neutron production rate and the neutron loss rate.

The name "multiplication factor" is used because k represents the ratio between the neutron density for one generation of neutrons, divided by the neutron density for the previous generation. This needs some explaining.

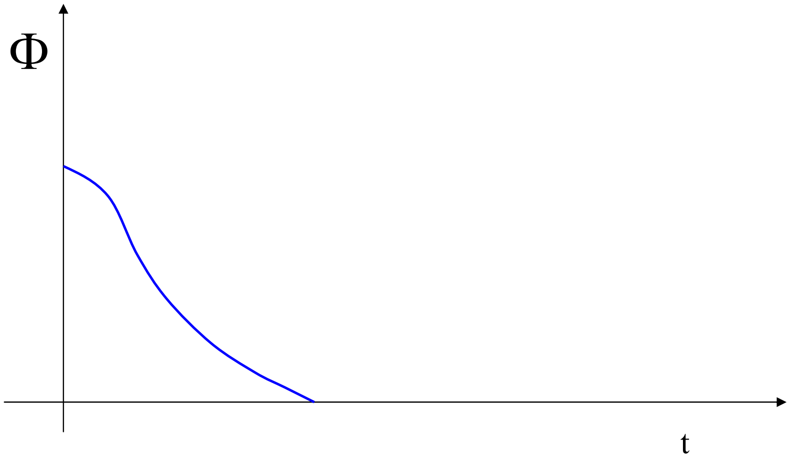
Consider a bare infinite homogeneous reactor. Initially there are no neutrons present.

Now, assume some neutrons, with density n_0 are introduced in the reactor. Let's call these "generation 0" neutrons. These neutrons will fly around, producing a flux $\Phi_0(t) = n_0(t)\mathrm{v}$ which will decrease as the neutrons are absorbed, until all neutrons are eventually absorbed.

The time dependence of the zero-generation neutrons looks something like this:



The flux, has a similar shape



As these zeroth-generation neutrons are absorbed, some of them produce fissions. We consider the neutrons born out of these fissions first generation neutrons. They are produced at a rate:

$$\nu\Sigma_f\Phi_0(t)$$

and are absorbed at a rate

$$\Sigma_a\Phi_1(t)$$

Overall, the number of first-generation neutrons that are produced per unit volume is:

$$n_1 = \int_0^\infty \nu\Sigma_f\Phi_0(t)dt = \nu\Sigma_f\int_0^\infty \Phi_0(t)dt = \nu\Sigma_f\psi_0$$

The total number of absorptions of first-generation neutrons is:

$$\int_0^\infty \Sigma_a\Phi_1(t)dt = \Sigma_a\int_0^\infty \Phi_1(t)dt = \Sigma_a\psi_1$$

Since, in the end, all first-generation neutrons get absorbed, we have:

$$\Sigma_a \psi_1 = \nu \Sigma_f \psi_0$$

which yields:

$$\psi_1 = \frac{\nu \Sigma_f}{\Sigma_a} \psi_0 = k_\infty \psi_0$$

The first-generation neutrons, in turn, produce second generation neutrons. Their number is:

$$n_2 = \nu \Sigma_f \psi_1 = \nu \Sigma_f k_\infty \psi_0 = k_\infty n_1$$

The process continues:

$$n_3 = k_\infty n_2$$

and so on.

The number of neutrons in each generation is equal to the number in the previous generation multiplied by k_∞. Hence the name *multiplication factor*.

Infinite Homogeneous Reactor (Two-Group Diffusion Approximation)

Diffusion equations:

$$-D_1\nabla^2\Phi_1 + \Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2$$
$$-D_2\nabla^2\Phi_2 - \Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = 0$$

Because the reactor is infinite and the flux (both fast and thermal) is constant in space, we have:

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2$$
$$-\Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = 0$$

Attempt to solve the system:

Group 2 equation yields:

$$\Phi_2 = \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\Phi_1$$

Substituting into the group 1 equation, we obtain:

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\Phi_1$$

Obviously, the above is only satisfied if:

$$\Sigma_{a1} + \Sigma_{s1\to 2} = \left(\nu\Sigma_{f1} + \nu\Sigma_{f2} \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}} \right) \frac{1}{k}$$

which may not always be the case. This means that unless the above is satisfied, we cannot have a steady-state solution to our diffusion equations.

To force the system of equations to have a (steady-state) solution, we resort to the same trick as before: use a "fudge factor" 1/k that multiplies fission productions.

Thus, our equations become:

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \frac{1}{k_\infty}\left(\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2 \right)$$

$$-\Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = 0$$

And, by substituting $\Phi_2 = \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\Phi_1$ into the fast-group equation, we obtain:

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \frac{1}{k}\left(\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2} \frac{\Sigma_{s1\to 2}}{\Sigma_{a2}}\Phi_1 \right)$$

Dividing by the flux, we obtain:

$$\Sigma_{a1} + \Sigma_{s1\rightarrow 2} = \frac{1}{k_\infty}\left(\nu\Sigma_{f1} + \nu\Sigma_{f2}\frac{\Sigma_{s1\rightarrow 2}}{\Sigma_{a2}}\right)$$

We can now solve for k_∞ .

$$k_\infty = \frac{\nu\Sigma_{f1} + \nu\Sigma_{f2}\frac{\Sigma_{s1\rightarrow 2}}{\Sigma_{a2}}}{\Sigma_{a1} + \Sigma_{s1\rightarrow 2}}$$

Choosing k_∞ to have the above value ensures the system admits a solution.

That solution is

$$\Phi_2 = \frac{\Sigma_{s1\rightarrow 2}}{\Sigma_{a2}}\Phi_1$$

We cannot find the fast flux explicitly.

A close look at the system of equations

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \frac{1}{k_\infty}\left(\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2\right)$$

$$-\Sigma_{s1\to 2}\Phi_1 + \Sigma_{a2}\Phi_2 = 0$$

reveals that it is a homogeneous system of linear equations which defines an eigenvalue/eigenvector problem. The eigenvalue is $1/k_\infty$ and, as expected, the eigenvector can only be determined up to a multiplicative constant which, in our solution, is Φ_1.

k_∞ can, in the two-group case be interpreted in three different ways:

1. the eigenvalue that allows the system of equations to have a solution
2. the ratio of productions over losses
3. the factor by which the number of neutrons gets multiplied from one generation to the next

Criticality

K<1 - Subcritical

- Number of neutrons decreases form one generation to the next
- Rate of neutron production smaller than rate of neutron loss

K=1 - Critical

- Number of neutrons stays constant form one generation to the next
- Rate of neutron production equals rate of neutron loss

K>1 - Supercritical

- Number of neutrons increases form one generation to the next
- Rate of neutron production larger than rate of neutron loss

Neutron Life Cycle, Four Factor Formula, Six Factor Formula

The Four-Factor Formula

Let us look at the group 1 equation in the two-group approximation.

$$\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1 = \frac{1}{k_\infty}\left(\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2\right)$$

Solving for the multiplication factor, we obtain:

$$k_\infty = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1}$$

The above can be processed as follows:

$$k_\infty = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1} = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1}\frac{\nu\Sigma_{f2}\Phi_2}{\nu\Sigma_{f2}\Phi_2} =$$

$$= \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\nu\Sigma_{f2}\Phi_2}\frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to 2}\Phi_1}$$

By making the notation:

$$\varepsilon = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\nu\Sigma_{f2}\Phi_2}$$

We obtain:

$$k_\infty = \varepsilon \frac{\nu \Sigma_{f2} \Phi_2}{\Sigma_{a1} \Phi_1 + \Sigma_{s1\to 2} \Phi_1}$$

We can continue the processing:

$$k_\infty = \varepsilon \frac{\nu \Sigma_{f2} \Phi_2}{\Sigma_{a1} \Phi_1 + \Sigma_{s1\to 2} \Phi_1} \frac{\Sigma_{a2} \Phi_2}{\Sigma_{a2} \Phi_2} = \varepsilon \frac{\Sigma_{a2} \Phi_2}{\Sigma_{a1} \Phi_1 + \Sigma_{s1\to 2} \Phi_1} \frac{\nu \Sigma_{f2} \Phi_2}{\Sigma_{a2} \Phi_2}$$

Denoting:

$$p = \frac{\Sigma_{a2} \Phi_2}{\Sigma_{a1} \Phi_1 + \Sigma_{s1\to 2} \Phi_1}$$

We have:

$$k_\infty = \varepsilon p \frac{\nu \Sigma_{f2} \Phi_2}{\Sigma_{a2} \Phi_2}$$

We can, moreover divide the thermal absorption cross section into the absorption cross section for fuel, and the one for moderator.

$$\Sigma_{a2} = \Sigma_{a2}^{\text{fuel}} + \Sigma_{a2}^{\text{moderator}}$$

With this, we can rewrite the formula for the multiplication factor as follows:

$$k_\infty = \varepsilon p \frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a2}\Phi_2} \frac{\Sigma_{a2}^{\text{fuel}}\Phi_2}{\Sigma_{a2}^{\text{fuel}}\Phi_2} = \varepsilon p \frac{\Sigma_{a2}^{\text{fuel}}\Phi_2}{\Sigma_{a2}\Phi_2} \frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a2}^{\text{fuel}}\Phi_2}$$

Denoting:

$$f = \frac{\Sigma_{a2}^{\text{fuel}}\Phi_2}{\Sigma_{a2}\Phi_2}$$

and

$$\eta = \frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a2}^{\text{fuel}}\Phi_2}$$

We obtain:

$$k_\infty = \varepsilon p f \eta$$

This is known as the *four factor formula*.

The names and interpretation of the factors are as follows:

Fast fission factor

$$\varepsilon = \frac{\nu\Sigma_{f1}\Phi_1 + \nu\Sigma_{f2}\Phi_2}{\nu\Sigma_{f2}\Phi_2} = \frac{total\ \ fission\ rate}{thermal\ \ fission\ rate}$$

Resonance escape probability

$$p = \frac{\Sigma_{a2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to2}\Phi_1} \ = \frac{\Sigma_{s1\to2}\Phi_2}{\Sigma_{a1}\Phi_1 + \Sigma_{s1\to2}\Phi_1}$$
$$= \frac{rate\ of\ \ slowing\ down}{rate\ of\ \ slowing\ down + absorptions}$$

Thermal utilization factor

$$f = \frac{\Sigma_{a2}^{\text{fuel}}\Phi_2}{\Sigma_{a2}\Phi_2} = \frac{rate\ of\ \ thermal\ \ absorption\ \ in\ \ fuel}{total\ \ rate\ \ of\ \ thermal\ \ absorption\ s}$$

η (number of neutrons produced per neutron absorbed in fuel)

$$\eta = \frac{\nu\Sigma_{f2}\Phi_2}{\Sigma_{a2}^{\text{fuel}}\Phi_2}$$
$$= \frac{rate\ of\ neutron\ production\ through\ thermal\ \ fission}{rate\ of\ thermal\ absorption}$$

Six Factor Formula

For a finite reactor, in addition to the processes we studied above, fast neutrons, as well as thermal neutrons can leak out of the reactor.

We define the following two factors to account for the leakage:

α_f = fast non-leakage probability
α_t = thermal non-leakage probability

Our expression for k then becomes the **six**-factor formula:

$$k_{eff} = \varepsilon p f \eta \alpha_f \alpha_t$$

One-Group Treatment of Finite Reactors

Diffusion Equation

$$D\nabla^2\Phi - \Sigma_a\Phi + \frac{1}{k}\mathrm{v}\Sigma_\mathrm{f}\Phi = 0$$

$$\nabla^2\Phi + \frac{1}{D}\left(-\Sigma_a + \frac{1}{k}\nu\Sigma_\mathrm{f}\right)\Phi = 0$$

Notation:
(<u>B^2 is called **Buckling**</u>)

$$B^2 = \frac{1}{D}\left(-\Sigma_a + \frac{1}{k}\nu\Sigma_\mathrm{f}\right)$$

The equation can be rewritten:

$$\nabla^2\Phi + B^2\Phi = 0$$

B depends on k. It turns out that B cannot take just any value. It has to be equal to the value imposed by the geometry, called the geometrical buckling.

$$B^2 = B_g^2$$

Then:

$$\frac{1}{D}\left(-\Sigma_a+\frac{1}{k}\nu\Sigma_{\mathrm{f}}\right)=B_g^2$$

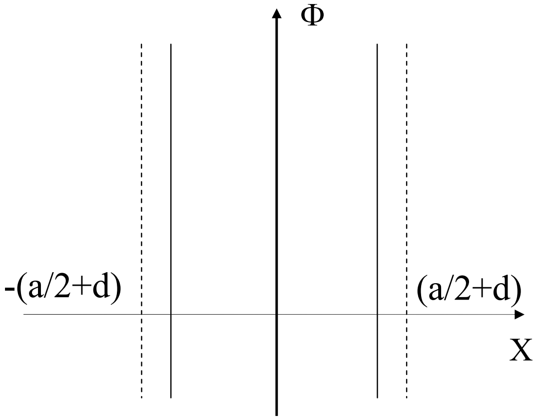
offers an equation for k.

$$k=\frac{\nu\Sigma_{\mathrm{f}}}{B_g^2 D+\Sigma_a}$$

Where $B_g^2 D$ is the leakage.

Things will become clearer by showing an example.

Infinite Slab Reactor



$$\Phi\left(\frac{\widetilde{a}}{2}\right)=0$$

where

$$\widetilde{a}=a+2d$$

We then have:

$$\frac{d^2\Phi}{dx^2} + B^2\Phi = 0$$

Boundary conditions:

$$\Phi\left(\frac{\tilde{a}}{2}\right) = \Phi\left(\frac{-\tilde{a}}{2}\right) = 0$$

The symmetry of problem implies:

$$\left.\frac{d\Phi}{dx}\right|_{x=0} = 0$$

General Solution:

$$\Phi(x) = A\cos Bx + C\sin Bx$$

$$\left.\frac{d\Phi(x)}{dx}\right|_{x=0} = \left[-AB\sin Bx + CB\cos Bx\right] =$$

$$= CB = 0 \Rightarrow C = 0$$

Hence:

$$\Phi(x) = A\cos Bx$$

Vacuum B.C.

$$\Phi\left(\frac{\tilde{a}}{2}\right) = A\cos\left(\frac{B\tilde{a}}{2}\right) = 0$$

Implies:

$$\cos\left(\frac{B\tilde{a}}{2}\right) = 0$$

$$\frac{B\tilde{a}}{2} = \frac{\pi}{2} + k\pi$$

$$B\tilde{a} = \pi + 2k\pi = (2k+1)\pi = n\pi$$

Yields:

$$B_n = \frac{n\pi}{\tilde{a}}\ ; \qquad B_1 = \frac{\pi}{\tilde{a}}$$

Fundamental solution

$$\Phi(x) = A\cos B_1 x = A\cos\left(\frac{\pi x}{\tilde{a}}\right)$$

B_1 is the geometrical buckling

A cannot be determined from the diffusion equation. It can be determined from the condition on the reactor power.

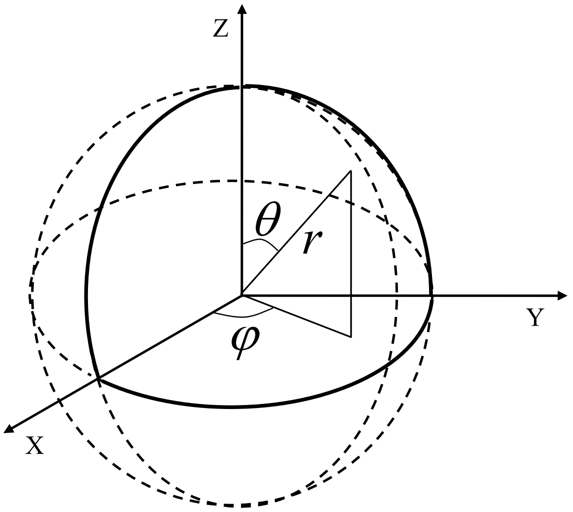
$$P = E_R \Sigma_f \int_{-a/2}^{a/2} \Phi(x)dx$$

$$P = \frac{2\tilde{a} E_R \Sigma_f A \sin\left(\frac{\pi a}{2\tilde{a}}\right)}{\pi}$$

$$\Phi(x) = \frac{\pi P}{2 a E_R \Sigma_f} \cos\left(\frac{\pi x}{a}\right)$$

$$k = \frac{\nu \Sigma_{\mathrm{f}}}{B_g^2 D + \Sigma_a} = \frac{\nu \Sigma_{\mathrm{f}}}{\left(\frac{\pi}{\tilde{a}}\right)^2 D + \Sigma_a}$$

Spherical Reactor



$$w = \Phi r \text{ (change of variable)}$$

$$\frac{d^2 w}{dr^2} + B^2 w = 0$$

$$w(r) = A \sin Br + C \cos Br$$

$$\Phi(r) = \frac{w}{r}$$

We have, in sequence:

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\Phi}{dr}+B^2\Phi=0$$

$$\Phi = A\frac{\sin Br}{r}+C\frac{\cos Br}{r}$$

Because the flux has to be finite at r=0, we have:

$$C=0$$

$$\Phi = A\frac{\sin Br}{r}$$

B.C. $\Phi(\widetilde{R})=0$
$\widetilde{R}=R+d$

$$B_1^2=\left(\frac{\pi}{\widetilde{R}}\right)^2$$

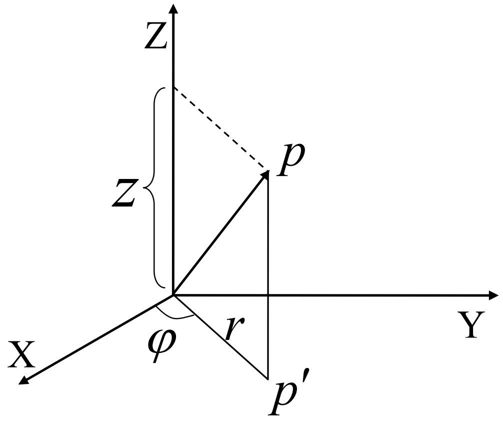
$$\Phi = A\frac{\sin {\pi r}/{\widetilde{R}}}{r}$$

The total power can be used to find A.

$$P=E_R\Sigma_f\int\Phi(r)dV \qquad dV=4\pi r^2dr$$

$$\Phi=\frac{P}{4E_R\Sigma_f R^2}\frac{\sin\left({\pi r}/{\widetilde{R}}\right)}{r}$$

Infinite Cylinder



Cylindrical coordinates

We have in sequence:

$$\frac{1}{r}\frac{d}{dr}r\frac{d\Phi}{dr}+B^2\Phi=0$$

$$\frac{d^2\Phi}{dr^2}+\frac{1}{r}\frac{d\Phi}{dr}+B^2\Phi=0$$

Bessel's Equation:

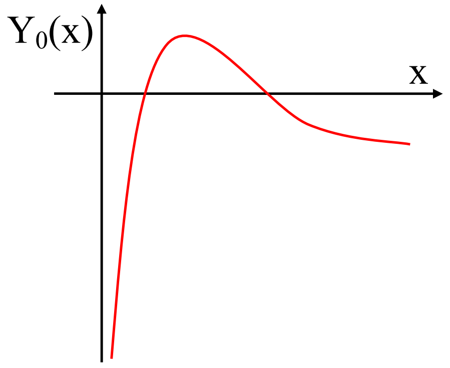
$$\frac{d^2\Phi}{dr^2}+\frac{1}{r}\frac{d\Phi}{dr}+\left(B^2-\frac{m^2}{r^2}\right)\Phi=0$$

Our equation is Bessel's equation for m=0.

Solution: Bessel functions of first and second kind:

$$\Phi = AJ_0(Br) + CY_0(Br)$$

J – Bessel Function
Y- Modified Bessel Function

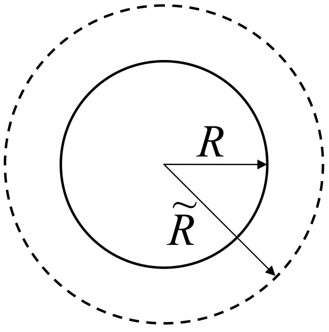


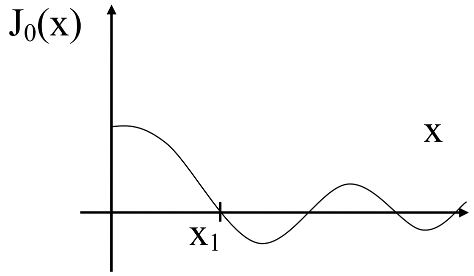
Y_0 infinite at origin hence C=0

$$\Phi = AJ_0(Br)$$

B.C.

$$\Phi(\widetilde{R}) = AJ_0(B\widetilde{R}) = 0$$





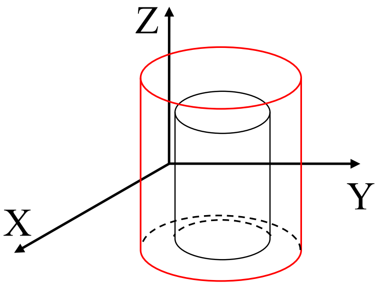
Final Solution

$$B\widetilde{R} = x_1 \Rightarrow B = \frac{x_1}{\widetilde{R}}$$

$$B_1^2 = \left(\frac{x_1}{\widetilde{R}}\right)^2 = \left(\frac{2.405}{\widetilde{R}}\right)^2$$

$$\Phi = AJ_0\left(\frac{2.405r}{\widetilde{R}}\right)$$

Finite Cylinder



$$\frac{d^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0$$

B.C.

$$\Phi(\widetilde{R}, z) = 0$$

$$\Phi(r, \frac{\widetilde{H}}{2}) = 0$$

Separation of Variables

$$\Phi(r, z) = R(r)Z(z)$$

$$\frac{\partial Z}{\partial r} = 0, \qquad \frac{\partial R}{\partial r} = 0$$

$$Z\frac{1}{r}\frac{d}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + R\frac{\partial^2 Z}{\partial z^2} + B^2 RZ = 0$$

$$\frac{1}{R}\frac{1}{r}\frac{d}{\partial r}r\frac{\partial R}{\partial r} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -B^2$$

$$\frac{1}{R}\frac{1}{r}\frac{d}{\partial r}r\frac{\partial R}{\partial r} = -B_r^2; \qquad R = AJ_0(B_r r)$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -B_z^2; \qquad Z = A\cos(B_z z)$$

Solution:

$$\Phi(r,z) = AJ_0\left(\frac{2.405r}{\widetilde{R}}\right)\cos\frac{\pi z}{\widetilde{H}}$$

$$\Phi(r,z) = AJ_0(B_r r)\cos(B_z z)$$