Nuclear Reactor Kinetics

Nomenclature

Time-Dependent Phenomena

- Short Time Phenomena (ms, s)
  - accidents
  - experiments
  - startup
- Medium Time Phenomena (hrs, days)
  - fission product poisoning
  - Xe
  - Sm
- Long Time Phenomena (months, years)
  - fuel burnup with consequent change in composition

Time-Dependent Phenomena

- No feedback (approximation)
  - Changes in flux level do not induce changes in the absorption or production properties of the reactor.
- Feedback
  - Changes in flux level do induce changes in the absorption or production properties of the reactor.

General Mathematical Formulation

Energy-and-time-dependent neutron balance equation with feedback in the diffusion approximation

$$\chi_p(E)\int_{E'=0}^{\infty}\nu_p\Sigma_f(\Phi,\vec{r},E',t)\Phi(\vec{r},E',t)dE'+\nabla\cdot\left(D(\Phi,\vec{r},E,t)\nabla\Phi(\Phi,\vec{r},E,t)\right)-$$

$$\Sigma_a(\Phi,\vec{r},E,t)\Phi(\vec{r},E,t)+\int_{E'=0}^{\infty}\Sigma_s(\Phi,\vec{r},E'\rightarrow E,t)\Phi(\vec{r},E',t)dE'+$$

$$S_d(\Phi,\vec{r},E,t)+S_e(\Phi,\vec{r},E,t)=\frac{1}{\overline{\mathrm{v}}}\frac{\partial\Phi(\vec{r},E,t)}{\partial t}$$

Throughout the course will use progressively more accurate approximations to solve the above.

Fission Facts

Fission Mechanism (simplified)

Fission occurs through the formation of a compound nucleus which, in turn, can decay very rapidly in several different ways.

$$ {}_{0}^{1}n + {}_{Z_X}^{A_X}X \rightarrow {}_{Z_X}^{A_X+1}X' $$

$$ {}_{Z_X}^{A_X+1}X' \begin{cases} \rightarrow \; A' + B' + \gamma & \text{(mode 1 - prompt } \gamma\text{)} \\ \rightarrow \; A' + B' + \tilde{\nu}_p n & \text{(mode 2 - prompt n)} \end{cases} $$

$$ \nu_p = 2 - 3 $$

Both A' and B' can be stable or further decay in several possible modes:

$$
A' \begin{cases} \longrightarrow A & (A' \text{ was stable}) \\ \xrightarrow{\lambda} A + \gamma & (\text{mode } 1) \\ \xrightarrow{\lambda} A + {}^{0}_{-1}\beta_{HE} & (\text{mode } 3) \\ \xrightarrow{\lambda} A'' + {}^{0}_{-1}\beta_{LE} & (\text{mode } 4) \\ \quad \downarrow (\text{fast}) & \\ \quad A + n & (\text{delayed neutron}) \end{cases}
$$

If A' decays according to mode 4, then it is called a *precursor* and A'' is called an *emitter*.

We cannot predict in advance which nuclei will be precursors, but we can predict, on the average how many will do so. This number is equal to the number of delayed neutrons emitted, called the *delayed neutron yield*.

$$\nu_d = \frac{\#\text{ of delayed neutrons}}{\#\text{ of fissions}}$$

We cannot predict how many prompt neutrons will be emitted in each reaction either. But we can predict how many will be produced on the average. This is called the *prompt neutron yield*.

$$\nu_p = \frac{\#\text{ of prompt neutrons}}{\#\text{ of fissions}}$$

On the average, the fission reaction can be written:

$$n + X \rightarrow A + B + \nu_p n_p + \nu_d n_d + \mu e + \gamma$$

The *total neutron yield* is defined as:

$$\nu = \nu_d + \nu_p$$

The delayed neutron fraction is:

$$\beta = \frac{\nu_d}{\nu}$$

Delayed Neutrons

Are emitted by emitters which result from the beta decay of precursors.

There are 6 precursor (delayed neutron) groups, based on their half-life.

TABLE 3.5 DELAYED NEUTRON DATA FOR THERMAL FISSION IN $^{235}U^{x}$

Group	Half-Life (sec)	Decay Constant (l_i, sec^{-1})	Energy (ke V)	Yield, Neutrons per Fission	Fraction (β_i)
1	55.72	0.0124	250	0.00052	0.000215
2	22.72	0.0305	560	0.00346	0.001424
3	6.22	0.111	405	0.00310	0.001274
4	2.30	0.301	450	0.00624	0.002568
5	0.610	1.14	—	0.00182	0.000748
6	0.230	3.01	—	0.00066	0.000273

Total yield: 0.0158

Total delayed fraction (β): 0.0065

Point Kinetics Equations

part 1

all neutrons emitted in a fission are assumed prompt

One-energy-group diffusion equation

- time-dependent diffusion (results from neutron balance)

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a \Phi(\vec{r},t)$$

- If sources are exactly equal to sinks, then the static equation results (no time dependence)

$$0 = \nu\Sigma_f \Phi(\vec{r}) + D\nabla^2\Phi(\vec{r}) - \Sigma_a \Phi(\vec{r}) \Leftrightarrow -D\nabla^2\Phi(\vec{r}) + \Sigma_a \Phi(\vec{r}) = \nu\Sigma_f \Phi(\vec{r})$$

One-energy-group diffusion equation

- To keep the static form of the diffusion equation even when the sources do not exactly equal the sinks, we introduced K, (multiplication factor) to artificially adjust the sources.

$$0 = \frac{1}{k}\nu\Sigma_f\Phi(\vec{r}) + D\nabla^2\Phi(\vec{r}) - \Sigma_a\Phi(\vec{r}) \Leftrightarrow -D\nabla^2\Phi(\vec{r}) + \Sigma_a\Phi(\vec{r}) = \frac{1}{k}\nu\Sigma_f\Phi(\vec{r})$$

- Now, we will not use k any more, but rather concentrate on the time-dependent equation

Time-dependent one-energy-group diffusion equation

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a\Phi(\vec{r},t)$$

- But let's remember:

$$\Phi = n\overline{\mathrm{v}} \Leftrightarrow n = \frac{\Phi}{\overline{\mathrm{v}}}$$

- so:

$$\frac{\partial n}{\partial t} = \frac{1}{\overline{\mathrm{v}}}\frac{\partial \Phi}{\partial t}$$

Time-dependence of the neutron flux and neutron density

- We can now write the time-dependent diffusion: in two separate ways
  - concentrate on the flux

$$\frac{1}{\overline{v}}\frac{\partial \Phi(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a \Phi(\vec{r},t)$$

  - concentrate on the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \overline{v} n(\vec{r},t) + D\nabla^2 \overline{v} n(\vec{r},t) - \Sigma_a \overline{v} n(\vec{r},t)$$

Some assumptions

- Static one-energy-group diffusion equation for a critical reactor

$$-D\nabla^2\Phi(\vec{r}) + \Sigma_a\Phi(\vec{r}) = \nu\Sigma_f\Phi(\vec{r})$$

- It can be rewritten as:

$$\nabla^2\Phi(\vec{r}) + B^2\Phi(\vec{r}) = 0$$

- Where

$$\frac{\nu\Sigma_f - \Sigma_a}{D} = B^2$$

Some assumptions

- Assume that the equation satisfied by the time-independent flux in a critical reactor is also satisfied, at any time t, by the time-dependent flux in a non-critical reactor.

$$\nabla^2 \Phi(\vec{r}) + B^2 \Phi(\vec{r}) = 0$$

$$\nabla^2 \Phi(\vec{r},t) + B^2 \Phi(\vec{r},t) = 0 \Leftrightarrow \nabla^2 \Phi(\vec{r},t) = -B^2 \Phi(\vec{r},t)$$

- This is equivalent to assuming that the spatial shape of the flux does not change with time

Back to the time-dependence of the neutron flux and neutron density

- We can now write the time-dependent diffusion
  - for the flux

$$\frac{1}{\overline{\mathrm{v}}}\frac{\partial\Phi(\vec{r},t)}{\partial t}=\nu\Sigma_f\Phi(\vec{r},t)-DB^2\Phi(\vec{r},t)-\Sigma_a\Phi(\vec{r},t)$$

  - for the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t}=\nu\Sigma_f\overline{\mathrm{v}}n(\vec{r},t)-DB^2\overline{\mathrm{v}}n(\vec{r},t)-\Sigma_a\overline{\mathrm{v}}n(\vec{r},t)$$

Time-dependence of the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t} = \left(\nu\Sigma_f \overline{\mathrm{v}} - DB^2\overline{\mathrm{v}} - \Sigma_a \overline{\mathrm{v}}\right) n(\vec{r},t) \Leftrightarrow \frac{\partial n(\vec{r},t)}{\partial t} = \alpha n(\vec{r},t)$$

- where

$$\alpha = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\overline{\mathrm{v}}$$

Time-dependence of the neutron density

$$\frac{\partial n(\vec{r},t)}{\partial t} = \alpha n(\vec{r},t)$$

- Integrating over the entire reactor we obtain:

$$\int_V \frac{\partial n(\vec{r},t)}{\partial t} d^3\vec{r} = \int_V \alpha n(\vec{r},t) d^3\vec{r}$$

$$\frac{d}{dt} \int_V n(\vec{r},t) d^3\vec{r} = \alpha \int_V n(\vec{r},t) d^3\vec{r}$$

Time-dependence of the total neutron population

- Total neutron population

$$n(t) = \int_V n(\vec{r},t)\,d^3\vec{r}$$

- Equation governing the time behaviour of the total neutron population

$$\frac{d}{dt}n(t) = \alpha n(t) \Leftrightarrow \dot{n}(t) = \alpha n(t)$$

- solution

$$n(t) = n_0 e^{\alpha t}$$

Time dependence of the neutron flux

$$\frac{1}{\overline{\mathrm{v}}}\frac{\partial \Phi(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) - DB^2\Phi(\vec{r},t) - \Sigma_a \Phi(\vec{r},t)$$

$$\frac{1}{\overline{\mathrm{v}}}\frac{\partial \Phi(\vec{r},t)}{\partial t} = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\Phi(\vec{r},t)$$

$$\frac{\partial \Phi(\vec{r},t)}{\partial t} = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\overline{\mathrm{v}}\Phi(\vec{r},t) \Leftrightarrow \frac{\partial \Phi(\vec{r},t)}{\partial t} = \alpha\Phi(\vec{r},t)$$

- The results are analogous to those for the neutron density.

Time dependence of the neutron flux

- Integrating over the volume of the reactor:

$$\int_V \frac{\partial \Phi(\vec{r},t)}{\partial t} d^3\vec{r} = \int_V \alpha \Phi(\vec{r},t) d^3\vec{r}$$

$$\frac{d}{dt}\int_V \Phi(\vec{r},t) d^3\vec{r} = \alpha \int_V \Phi(\vec{r},t) d^3\vec{r}$$

$$\frac{d}{dt}\hat{\phi}(t) = \alpha\hat{\phi}(t) \Leftrightarrow \dot{\hat{\phi}}(t) = \alpha\hat{\phi}(t)$$

- where:

$$\hat{\phi}(t) = \int_V \Phi(\vec{r},t) d^3\vec{r}$$

Observations

- The total neutron population and the volume integrated flux obey the same equation.
- The relation between the volume integrated flux and the total neutron population is the same as that between the flux and neutron density.

$$\Phi(\vec{r},t) = n(\vec{r},t)\overline{\mathrm{v}}$$

$$\int_V \Phi(\vec{r},t)d^3\vec{r} = \overline{\mathrm{v}}\int_V n(\vec{r},t)d^3\vec{r} \Leftrightarrow \hat{\phi}(t) = n(t)\overline{\mathrm{v}}$$

Point Kinetics Equation without Delayed Neutrons

- Just a special way of arranging the coefficients.
- Usually written for the neutron population, but similar equation can be written for the volume-integrated flux.

Point Kinetics Equation without Delayed Neutrons

- Multiplication constant $k_{eff} = \frac{\nu\Sigma_f}{\Sigma_a + DB^2}$
- Reactivity $\rho = \frac{k-1}{k} = 1 - \frac{1}{k}$
- we can write:

$$\alpha = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\overline{\mathrm{v}} = \nu\Sigma_f\overline{\mathrm{v}}\frac{\nu\Sigma_f - DB^2 - \Sigma_a}{\nu\Sigma_f} =$$

$$\nu\Sigma_f\overline{\mathrm{v}}\left(\frac{\nu\Sigma_f}{\nu\Sigma_f} - \frac{DB^2 + \Sigma_a}{\nu\Sigma_f}\right) = \nu\Sigma_f\overline{\mathrm{v}}\left(1 - \frac{1}{k_{eff}}\right) = \nu\Sigma_f\overline{\mathrm{v}}\rho$$

Point Kinetics Equation without Delayed Neutrons

- Notation: $$\Lambda = \frac{1}{\nu \Sigma_f \bar{v}}$$

- It follows that:

$$\alpha = \frac{\rho}{\Lambda}$$

- The equation for the neutron population can then be written

$$\frac{dn(t)}{dt} = \frac{\rho}{\Lambda} n(t)$$ = Point kinetics eq. w/o dn

Point Kinetics Equation without Delayed Neutrons

- A similar equation can be written for the volume-integrated flux.

$$\frac{d\hat{\phi}(t)}{dt} = \frac{\rho}{\Lambda}\hat{\phi}(t)$$

- Alternative processing:

$$\alpha = \left(\nu\Sigma_f - DB^2 - \Sigma_a\right)\overline{\mathrm{v}} = \left(DB^2 + \Sigma_a\right)\overline{\mathrm{v}}\,\frac{\nu\Sigma_f - DB^2 - \Sigma_a}{DB^2 + \Sigma_a} =$$

$$\left(DB^2 + \Sigma_a\right)\overline{\mathrm{v}}\left(\frac{\nu\Sigma_f}{DB^2 + \Sigma_a} - \frac{DB^2 + \Sigma_a}{DB^2 + \Sigma_a}\right) = \left(DB^2 + \Sigma_a\right)\overline{\mathrm{v}}\left(k - 1\right)$$

Point Kinetics Equation without Delayed Neutrons

- New notation

$$\ell = \frac{1}{\left(DB^2 + \Sigma_a\right)\overline{v}}$$

- With the new notation the point kinetics eq. can be written (a less common form):

$$\frac{dn(t)}{dt} = \frac{k-1}{\ell} n(t)$$

- and, for the flux:

$$\frac{d\hat{\phi}(t)}{dt} = \frac{k-1}{\ell} \hat{\phi}(t)$$

Point kinetics equation(s)

- Nomenclature – called point-kinetics because the reactor is reduced to a point – no accounting for spatial or energy dependence.
- Can be derived starting from a more general, space and energy dependent, flux.

Names and interpretations of symbols

- Neutron generation time

$$\Lambda = \frac{1}{\bar{\mathrm{v}}\nu\Sigma_f}$$

- Interpretations
  - Average time between two neutron births in successive generations
  - Time it would take to generate the current number of neutrons at the current generation rate.
  - Average “age” of neutrons in the reactor. (Note that this is a time, and not the Fermi age).

Names and interpretations of symbols

- Neutron life time

$$\ell = \frac{1}{\bar{v}} \frac{1}{\Sigma_a + DB^2}$$

- For an infinite reactor: $\ell_\infty = \frac{1}{\bar{v}} \frac{1}{\Sigma_a}$
- Interpretations
  - average time between the birth and death of a neutron
  - Time necessary to lose all the neutrons in the reactor at the current loss rate.
  - Average life expectancy for neutrons in the reactor.

Point Kinetics Equations

part 2

Accounting for Delayed Neutrons

Point Kinetics with Only One Delayed Neutron Group

- We make the same assumptions about the buckling staying constant as in the case with no delayed neutrons.
- We write directly the equation for the entire reactor (volume-integrated quantities)
- Some neutrons are emitted directly from fission
- Some neutrons come from the decay of precursors.

Neutron Balance Equation for the Entire Reactor

- Sources
  - Prompt neutrons from fission

$$\int_V \nu_p \Sigma_f \Phi(\vec{r}) d^3\vec{r} = \nu_p \Sigma_f \int_V \Phi(\vec{r}) d^3\vec{r} = \nu_p \Sigma_f \hat{\phi} = (\nu - \nu_d)\Sigma_f \hat{\phi} = \nu(1-\beta)\Sigma_f \hat{\phi}$$

  - Delayed neutrons from the decay of precursors

$$\lambda \hat{C} \qquad (\hat{\mathrm{C}} = \text{total number of precursors in the reactor})$$

Neutron Balance Equation for the Entire Reactor

- Sinks
  - Absorption

$$\int_V \Sigma_a \Phi(\vec{r}) d^3\vec{r} = \Sigma_a \int_V \Phi(\vec{r}) d^3\vec{r} = \Sigma_a \hat{\phi}$$

  - Leakage

$$\int_V DB^2 \Phi(\vec{r}) d^3\vec{r} = DB^2 \int_V \Phi(\vec{r}) d^3\vec{r} = DB^2 \hat{\phi}$$

Precursor Balance Equation for the Entire Reactor

- Source

$$\int_V \nu_d \Sigma_f \Phi(\vec{r}) d^3\vec{r} = \nu_d \Sigma_f \int_V \Phi(\vec{r}) d^3\vec{r} = \nu_d \Sigma_f \hat{\phi} = \nu\beta\Sigma_f \hat{\phi}$$

- Sink

$$\lambda \hat{C}$$

Neutron and Precursor balance Equations

- Neutron Balance

$$\frac{dn(t)}{dt} = \nu_p \Sigma_f \hat{\phi} - \Sigma_a \hat{\phi} - DB^2 \hat{\phi} + \lambda \ \hat{C}$$

- Precursor Balance

$$\frac{d\hat{C}(t)}{dt} = \nu_d \Sigma_f \hat{\phi} - \lambda \hat{C}(t)$$

- We now have a system of two (coupled) differential equations.

Point Kinetics Equations with One Group of Delayed Neutrons

- Rearrange the first equation in a few steps

$$\frac{dn(t)}{dt} = \left[\nu(1-\beta)\Sigma_f - \Sigma_a - DB^2\right]\hat{\phi} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \nu\Sigma_f\,\frac{\left[\nu(1-\beta\ )\Sigma_f - \Sigma_a - DB^2\right]}{\nu\Sigma_f}\hat{\phi} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \bar{v}\,\nu\Sigma_f\left[\frac{\nu\Sigma_f - \Sigma_a - DB^2}{\nu\Sigma_f} - \frac{\beta\nu\Sigma_f}{\nu\Sigma_f}\right]\frac{\hat{\phi}}{\bar{\mathrm{v}}} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \bar{v}\,\nu\Sigma_f\left[1 - \frac{\Sigma_a + DB^2}{\nu\Sigma_f} - \beta\right]\frac{\hat{\phi}}{\bar{v}} + \lambda\,\hat{C}$$

Point Kinetics Equations with One Group of Delayed Neutrons

- Rearrange the second equation

$$\frac{d\hat{C}(t)}{dt} = \nu_d \Sigma_f \hat{\phi} - \lambda \hat{C}(t)$$

$$\frac{d\hat{C}(t)}{dt} = \nu \beta \Sigma_f \hat{\phi} - \lambda \hat{C}(t)$$

$$\frac{d\hat{C}(t)}{dt} = \beta \overline{\mathrm{v}} \nu \Sigma_f \frac{\hat{\phi}}{\overline{\mathrm{v}}} - \lambda \hat{C}(t)$$

Point Kinetics Equations with One Group of Delayed Neutrons

- Make the same notations and observations as for the case with no delayed neutrons

$$\Lambda = \frac{1}{\nu \Sigma_f \overline{\mathrm{v}}}$$

$$\hat{\phi}(t) = n(t)\overline{\mathrm{v}} \Leftrightarrow n(t) = \frac{\hat{\phi}(t)}{\overline{\mathrm{v}}}$$

$$k = \frac{\nu \Sigma_f}{\Sigma_a + DB^2}$$

$$\rho = 1 - \frac{1}{k} = 1 - \frac{\Sigma_a + DB^2}{\nu \Sigma_f}$$

Point Kinetics Equations with One Group of Delayed Neutrons

- Neutron Balance Equation

$$\frac{dn(t)}{dt} = \overline{\mathrm{v}}\,\nu\Sigma_f\left[1-\frac{\Sigma_a + DB^2}{\nu\Sigma_f} - \beta\right]\frac{\hat{\phi}}{\overline{\mathrm{v}}} + \lambda\,\hat{C}$$

$$\frac{dn(t)}{dt} = \frac{\rho-\beta}{\Lambda}n(t) + \lambda\,\hat{C}$$

- Precursor Balance Equation

$$\frac{d\hat{C}(t)}{dt} = \beta\overline{\mathrm{v}}\,\nu\Sigma_f\,\frac{\hat{\phi}}{\overline{\mathrm{v}}} - \lambda\hat{C}(t)$$

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda}n(t) - \lambda\hat{C}(t)$$

Point Kinetics Equations with One Group of Delayed Neutrons

- Final form of kinetics equations using the neutron population

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \lambda\,\hat{C}$$

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda \hat{C}(t)$$

Point Kinetics Equations with One Group of Delayed Neutrons

- Final form of the point kinetics equations using the volume-integrated flux

$$\hat{\phi}(t) = n(t)\overline{\mathrm{v}} \Leftrightarrow n(t) = \frac{\hat{\phi}(t)}{\overline{\mathrm{v}}}$$

$$\frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{\phi}(t) + \overline{\mathrm{v}}\lambda\,\hat{C} \Leftrightarrow \frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{\phi}(t) + \frac{1}{\Lambda\,\nu\Sigma_f}\lambda\,\hat{C}$$

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda}\frac{1}{\overline{\mathrm{v}}}\hat{\phi}(t) - \lambda\hat{C}(t) \Leftrightarrow \frac{d\hat{C}(t)}{dt} = \beta\nu\Sigma_f\hat{\phi}(t) - \lambda\hat{C}(t)$$

Point Kinetics Equations with Six Groups of Delayed Neutrons

- Equations using the neutron population (7 coupled differential equations)

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{k=1}^{6} \lambda_k \hat{C}_k$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda} n(t) - \lambda_k \hat{C}_k(t), \quad k = 1 \ldots 6$$

Point Kinetics Equations with Six Groups of Delayed Neutrons

- Equations using the volume-integrated flux (7 coupled differential equations)

$$\frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{\phi}(t) + \bar{\mathrm{v}}\sum_{k=1}^{6}\lambda_k \hat{C}_k \Leftrightarrow \frac{d\hat{\phi}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{\phi}(t) + \frac{1}{\Lambda \nu \Sigma_f}\sum_{k=1}^{6}\lambda_k \hat{C}_k$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda}\frac{1}{\bar{\mathrm{v}}}\hat{\phi}(t) - \lambda_k \hat{C}_k(t) \Leftrightarrow \frac{d\hat{C}_k(t)}{dt} = \beta_k \nu \Sigma_f \hat{\phi}(t) - \lambda_k \hat{C}_k(t)$$

External Source Point Kinetics Equations Example

A parallelepipedic homogeneous subcritical assembly has the following dimensions:

- Lx=2m
- Ly=2m
- Lz=4m

- material parameters
  - SF=0.0010 cm-1
  - SA=0.0026 cm-1
  - D=1 cm.
  - neutron yield = 2.5

Assumptions

- The extrapolated dimensions are approximately equal to the physical dimensions
- All neutrons are born as prompt neutrons.
- All neutrons move at 2200 m/s

A)

- **Calculate**: the neutron generation time, the neutron life time and the reactivity for this assembly.
-

B) The subcritical assembly is built in the proximity of a functioning nuclear reactor, and a thermal neutron conduit that can be opened and closed exists between the two. At t=0, the conduit is opened, so that the flux of neutrons _coming from the reactor_ is now distributed throughout the subcritical assembly in the same cosine shape as the static flux and equal to 10^{14} $n/cm^2/s$ at the center of the subcritical assembly. The conduit is kept open for 100 seconds.

For point B) Calculate:

- total (volume-integrated) neutron flux in the reactor
  - right after the conduit is opened
  - at 0.01 s
  - at 0.1 s
  - at 1.0 s
  - at 10 s
  - right **before** the conduit is closed at 100s
  - right **after** the conduit is closed at 100s
  - 0.01 s after the conduit is closed
  - 0.1 s after the conduit is closed
  - 1.0 s after the conduit is closed
  - 10.0 s after the conduit is closed
  - 100.0 s after the conduit is closed

Also for point B) Calculate:

- total neutron population in the reactor
  - right after the conduit is opened
  - at 0.01 s
  - at 0.1 s
  - at 1.0 s
  - at 10 s
  - right before the conduit is closed at 100s
  - right after the conduit is closed at 100s
  - 0.01 s after the conduit is closed
  - 0.1 s after the conduit is closed
  - 1.0 s after the conduit is closed
  - 10.0 s after the conduit is closed
  - 100.0 s after the conduit is closed

C) Assume now the neutrons move at an average speed of 22,000m/s.

- Calculate:
  - total neutron population in the reactor
    - right after the conduit is opened
    - at 0.01 s
    - at 0.1 s
    - at 1.0 s
    - at 10 s
    - right before the conduit is closed at 100s
    - right after the conduit is closed at 100s
    - 0.01 s after the conduit is closed
    - 0.1 s after the conduit is closed
    - 1.0 s after the conduit is closed
    - 10.0 s after the conduit is closed
    - 100.0 s after the conduit is closed

Example Solution

Calculate k_{eff} and ρ

$$B^2 = \left(\frac{\pi}{L_x}\right)^2 + \left(\frac{\pi}{L_y}\right)^2 + \left(\frac{\pi}{L_z}\right)^2 =$$

$$= \left(\frac{\pi}{2m}\right)^2 + \left(\frac{\pi}{2m}\right)^2 + \left(\frac{\pi}{4m}\right)^2 = 0.00055\left[cm^{-2}\right]$$

$$k_{eff} = \frac{\nu\Sigma_f}{\Sigma_a + DB^2} = \frac{0.001 \times 2.5}{0.0026 + 1 \times 0.00055} = 0.79$$

$$k < 1 \Rightarrow subcritical$$

$$\rho = 1 - \frac{1}{k} = -\frac{0.21}{0.79} = -0.266 \rightarrow -266\,mk$$

Note: $1mk \equiv 0.001$

$$\Lambda = \frac{1}{\bar{v}\,\nu\Sigma_f} = \frac{1}{2200\frac{m}{s} \times 100\frac{cm}{m} \times 2.5 \times 0.001cm^{-1}} = 0.0018(s)$$

$$\ell = \frac{1}{\bar{v}(\Sigma_a + DB^2)} = \frac{1}{2200\frac{m}{s} \times 100\frac{cm}{m} \times (0.0026cm^{-1} + 0.00055cm^{-1})} = 0.0014(s)$$

Setting up the differential equation

$$\dot{n}(t) = \frac{\rho}{\Lambda} n(t) + S$$

S=External source

Volumetric source (source per unit volume)

$$S(\vec{r}) = \Phi_{reactor} \cdot \nu\Sigma_f = 10^{14} \times 2.5 \times 0.001 =$$

$$= 2.5 \times 10^{11} \left[neutrons / cm^3 / s \right]$$

Total Source

$$S = \int_V S(\vec{r}) d^3\vec{r}$$

Assume that $S(\vec{r})$ follows the same shape as the neutron flux, that is:

$$S(x, y, z) = a \cdot \cos(B_x X) \cdot \cos(B_y Y) \cdot \cos(B_z Z)$$

$$\Phi_{reactor}(\vec{r}) = \Phi_{reactor}(x, y, z) = 10^{14} \cdot \cos(B_x x) \cdot \cos(B_y y) \cdot \cos(B_z z)$$

This is a reasonable assumption because:

$$S(\vec{r}) = \nu\Sigma_f \Phi_{reactor}(\vec{r}) = 2.5 \times 0.001 \times 10^{14} \cos(B_x x) \cdot \cos(B_y y) \cdot \cos(B_z z)$$

$$S(x, y, z) = 2.5 \times 10^{11} \cos(B_x x) \cdot \cos(B_y y) \cdot \cos(B_z z) =$$
$$a \times \cos(B_x x) \cdot \cos(B_y y) \cdot \cos(B_z z)$$

$$\Phi_{assembly}(\vec{r})$$

$$S = \int_V S(\vec{r}) d^3\vec{r}$$

$$S = \int_V S(\vec{r}) d^3\vec{r} = \iiint_{x,y,z} a \times \cos(B_x x) \cdot \cos(B_y y) \cdot \cos(B_z z) dxdydz$$

$$S = a \int_{-\frac{L_z}{2}}^{\frac{L_z}{2}} \left[\int_{-\frac{Ly}{2}}^{\frac{L_y}{2}} \left(\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x x) \cdot \cos(B_y y) \cdot \cos(B_z z) dx \right) dy \right] dz$$

Because $\cos(B_y y)\cdot\cos(B_z z)$ does not depend on x, we have for the innermost integral:

$$\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x x)\cdot\cos(B_y y)\cdot\cos(B_z z)dx = \cos(B_y y)\cdot\cos(B_z z)\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x x)dx$$

Substituting this back into the full triple integral, we have:

$$S = a\int_{-\frac{L_z}{2}}^{\frac{L_z}{2}}\left[\int_{-\frac{Ly}{2}}^{\frac{L_y}{2}}\left(\cos(B_y y)\cdot\cos(B_z z)\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}}\cos(B_x x)dx\right)dy\right]dz$$

And since $\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x x)dx$ doe not depend on either y or z, in can be taken out of those integrals

$$S = a\left[\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x x)dx\right] \times \int_{-\frac{L_z}{2}}^{\frac{L_z}{2}} \left[\int_{-\frac{Ly}{2}}^{\frac{L_y}{2}} \cos(B_y y) \cdot \cos(B_z z)dy\right]dz$$

Now, since $\cos(B_z z)$ does not depend on y, in can be pulled in front of the integral over y:

$$S = a\left[\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x x)dx\right] \times \int_{-\frac{L_z}{2}}^{\frac{L_z}{2}} \left[\cos(B_z z)\left(\int_{-\frac{Ly}{2}}^{\frac{L_y}{2}} \cos(B_y y)dy\right)\right]dz$$

And since $\left(\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} \cos(B_y y) dy \right)$ does not depend on z, it can be factored out of the z integral:

$$S = a \left[\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x x) dx \right] \times \int_{-\frac{L_z}{2}}^{\frac{L_z}{2}} \left[\cos(B_z z) \left(\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} \cos(B_y y) dy \right) \right] dz =$$

$$= a \left[\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x x) dx \right] \times \left[\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} \cos(B_y y) dy \right] \times \int_{-\frac{L_z}{2}}^{\frac{L_z}{2}} \cos(B_z z) dz$$

This last form is the product of three simple integrals, which we can easily process.

$$S = a \times \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x x)dx \times \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} \cos(B_x X)dx \times \int_{-\frac{L_z}{2}}^{\frac{L_z}{2}} \cos(B_z Z)dz$$

$$= a \times \frac{1}{B_x} \sin(B_x x)\Big|_{-\frac{L_x}{2}}^{+\frac{L_x}{2}} \times \frac{1}{B_y} \sin(B_y y)\Big|_{-\frac{L_y}{2}}^{+\frac{L_y}{2}} \times \frac{1}{B_z} \sin(B_z z)\Big|_{-\frac{L_z}{2}}^{+\frac{L_z}{2}} =$$

$$= \frac{a}{B_x B_y B_z} \cdot \left[\sin\left(\frac{B_x L_x}{2}\right) - \sin\left(\frac{-B_x L_x}{2}\right)\right] \times \left[\sin\left(\frac{B_y L_y}{2}\right) - \sin\left(\frac{-B_y L_y}{2}\right)\right] \times$$

$$\times \left[\sin\left(\frac{B_z L_z}{2}\right) - \sin\left(\frac{-B_z L_z}{2}\right)\right]$$

Remembering that the boundary conditions (B.C.) are $\Phi_b = 0$, we can immediately infer:

$$\Phi_b = 0 \Rightarrow \cos\left(\frac{B_x L_x}{2}\right) = 0 \Rightarrow B_x = \frac{\pi}{L_x} \Rightarrow \sin\left(\frac{B_x L_x}{2}\right) = 1$$

And same for y and z.

We hence find:

$$S = \frac{8a}{B_x B_y B_z} = \frac{8a}{\frac{\pi}{L_x}\frac{\pi}{L_y}\frac{\pi}{L_z}} = \frac{8}{\pi^3} a L_x L_y L_z =$$

$$= \frac{8}{\pi^3} 2.5 \times 10^{11} \left[\frac{fissions}{cm^3 \cdot s}\right] \times 200cm \times 200cm \times 400cm = 1.032 \times 10^{18} \left[fissions / s\right]$$

Now, the time-dependent neutron balance equation (for the entire assembly) in the presence of the external source is written:

$$\dot{n}(t) = \frac{\rho}{\Lambda} n(t) + S, \quad S = External\ source$$

We can now proceed to solving the differential equation

$$\dot{n}(t) - \frac{\rho}{\Lambda} n(t) - S$$

Apply the integrating factor $e^{-\frac{\rho t}{\Lambda}}$ yields in sequence:

$$\dot{n}(t)e^{-\frac{\rho t}{\Lambda}} - \frac{\rho}{\Lambda}e^{-\frac{\rho t}{\Lambda}} = Se^{-\frac{\rho t}{\Lambda}}$$

$$\frac{d}{dt}\left[n(t)e^{-\frac{\rho t}{\Lambda}}\right] = Se^{-\frac{\rho t}{\Lambda}}$$

$$n(t)e^{-\frac{\rho t}{\Lambda}} = \int Se^{-\frac{\rho t}{\Lambda}}dt$$

$$n(t)e^{-\frac{\rho t}{\Lambda}} - n(t_0)e^{-\frac{\rho t_0}{\Lambda}} = \int_{t_0}^{t} Se^{-\frac{\rho t'}{\Lambda}} dt'$$

$$n(t)e^{-\frac{\rho t}{\Lambda}} - n(t_0)e^{-\frac{\rho t_0}{\Lambda}} = -S \times \left[\frac{\Lambda}{\rho} e^{-\frac{\rho t}{\Lambda}} \Bigg|_{t_0}^{t} \right]$$

$$n(t) - n(t_0)e^{-\frac{\rho t_0}{\Lambda}} e^{\frac{\rho t}{\Lambda}} = -S \times \frac{\Lambda}{\rho} \left[e^{\frac{-\rho t}{\Lambda}} - e^{\frac{-\rho t_0}{\Lambda}} \right] e^{\frac{\rho t}{\Lambda}}$$

$$n(t) = n(t_0)e^{\frac{\rho(t-t_0)}{\Lambda}} - S \times \frac{\Lambda}{\rho} \left[1 - e^{\frac{\rho(t-t_0)}{\Lambda}} \right]$$

$$n(t)e^{-\frac{\rho t}{\Lambda}} = -S \cdot \frac{\Lambda}{\rho} e^{-\frac{\rho t}{\Lambda}} + C$$

$$n(t)e^{-\frac{\rho t}{\Lambda}} = -S \cdot \frac{\Lambda}{\rho} e^{-\frac{\rho t}{\Lambda}} + C$$

If the initial neutron population is zero, then:

$$n(t)e^{-\frac{\rho t}{\Lambda}} = \frac{S\Lambda}{\rho}\left(1 - e^{-\frac{\rho t}{\Lambda}}\right)$$

$$n(t) = \frac{S\Lambda}{\rho}\left(e^{\frac{\rho t}{\Lambda}} - 1\right)$$

Numerical substitutions yield:

$$S = 1.032 \times 10^{18} \left[fissions / s \right]$$

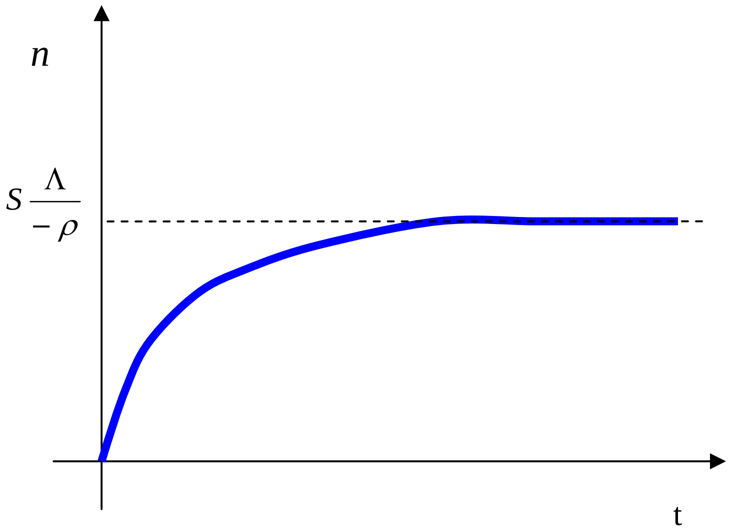
$$\rho = -0.266$$

$$\Lambda = 0.0018(s)$$

$$\ell = 0.0014(s)$$

$$n(t) = \frac{1.032 \times 10^{18} \times 0.0018}{-0.266} \left(e^{\frac{-0.266}{0.0018}t} - 1 \right)$$

$$n(t) = 6.983 \times 10^{15}\left(1 - e^{-147.78\times t}\right)$$



After a sufficient time, the neutron population reaches an equilibrium value:

$$n_{equil} = S\frac{\Lambda}{-\rho}$$

This is an important fact. As the reactivity is small and negative, the above shows, in fact, that the source is "multiplied" to obtain the neutron population. The smaller the absolute value of rho, that is the closer to critical the assembly is, the larger the neutron population. We will use this later in a method of measuring the reactivity, called the "source multiplication method".

The time dependence of the neutron population is, assuming the initial population (at the time the source was turned on) to be zero:

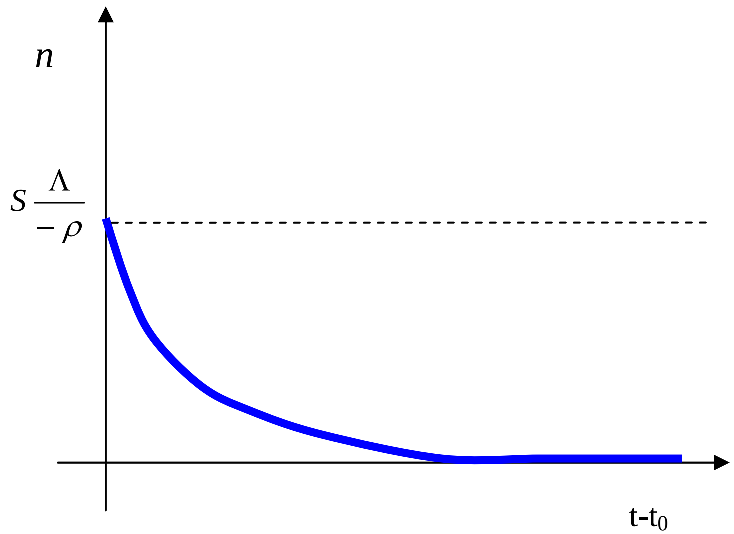
After the external source is turned off, the time dependence of the neutron population becomes:

$$n(t) = n(t_0)e^{\frac{\rho(t-t_0)}{\Lambda}}$$

Where t_0 is the time the external source was turned off.

Numerical substitutions yield:

$$n(t) = 6.983 \times 10^{15} \times e^{-147.78 \times (t-t_0)}$$



Let us now investigate what happens if the average speed is ten time larger larger (faster neutron spectrum)

$$\bar{v} = 22000\, m/s$$

$$\Lambda = \frac{1}{\bar{v}\,\nu\Sigma_f} = \frac{1}{22000\,\frac{m}{s} \times 100\,\frac{cm}{m} \times 2.5 \times 0.001 cm^{-1}} = 0.00018(s)$$ ten times smaller

After the source is turned on:

$n(t) = 6.983 \times 10^{14}\left(1 - e^{-1477.78 \times t}\right)$ Exponent coefficient ten times larger. So the time variation will be much faster. The smaller the generation time, the faster the variation of the neutron population. The equilibrium concentration is reached faster.

After the source is turned off:

$$n(t) = 6.983 \times 10^{14} \times e^{-1477.78 \times (t - t_0)}$$

The exponent is much larger, so the neutron population drops much faster than in the case of the lower average neutron speed.

Perturbation Theory

Perturbation Theory

- Consider a reactor with the following parameters (This is called the “reference”, or “unperturbed” reactor):

$$D_0(\vec{r})$$

$$\Sigma_{a0}(\vec{r})$$

$$\nu\Sigma_{f0}(\vec{r})$$

- K can be calculated by solving:

$$-D_0(\vec{r})\nabla^2\Phi_0(\vec{r}) + \Sigma_{a0}(\vec{r})\Phi_0(\vec{r}) = \frac{1}{k_0}\nu\Sigma_f(\vec{r})\Phi_0(\vec{r})$$

Perturbation Theory

- Consider now a different reactor, with the same shape, but different absorption and fission cross sections (This is called the "perturbed" reactor).

$$\Sigma_a(\vec{r}) = \Sigma_{a0}(\vec{r}) + \delta\Sigma_a(\vec{r})$$

$$\Sigma_f(\vec{r}) = \Sigma_{f0}(\vec{r}) + \delta\Sigma_f(\vec{r})$$

- K can be calculated by solving:

$$-D_0\nabla^2\Phi(\vec{r}) + \Sigma_a\Phi(\vec{r}) = \frac{1}{k}\nu\Sigma_f\Phi(\vec{r})$$

Perturbation Theory

- Assuming that

  $\left|\delta\Sigma_a(\vec{r})\right|$ is small

  $\left|\delta\Sigma_f(\vec{r})\right|$ is small

- we can find k by a simple formula, without having to solve the diffusion equation again
- NOTE:
  - $\delta\Sigma_a$ is called the perturbation in the absorption cross section
  - $\delta\Sigma_f$ is called the perturbation in the fission cross section

Perturbation Theory

- Perturbation Formula for finding k (no proof):

$$\frac{1}{k}-\frac{1}{k_0}\cong\frac{\int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV-\frac{1}{k_0}\int_V \nu\delta\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

- The formula is good even if the reactor is not homogeneous.

Perturbation Theory

- Q: What does "small" mean for a perturbation?
- A:

$$\int_V \left|\delta\Sigma_a(\vec{r})\right| dV << \int_V \Sigma_{a0}(\vec{r}) dV$$

$$\int_V \left|\delta\Sigma_f(\vec{r})\right| dV << \int_V \Sigma_{f0}(\vec{r}) dV$$

- A perturbation can be small in two ways:
  - a) $\left|\delta\Sigma_a(\vec{r})\right| << \Sigma_{a0}(\vec{r})$

    $\left|\delta\Sigma_f(\vec{r})\right| << \Sigma_{f0}(\vec{r})$
  - b) The perturbation only affects a small part of the reactor.

Reactivity Induced by a Perturbation in a Critical Reactor

- Reactor initially critical $k_0 = 1$
- Introduce a perturbation
- The new k can be found using the perturbation formula:

$$\frac{1}{k} - \frac{1}{k_0} \cong \frac{\int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV - \frac{1}{k_0}\int_V \nu\delta\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

Reactivity Induced by a Perturbation in a Critical Reactor

- Since $k_0 = 1$
- we can write:

$$\frac{1}{k} - 1 \cong \frac{\int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV - \int_V \nu\delta\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

$$1 - \frac{1}{k} \cong \frac{\int_V \nu\delta\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV - \int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

Reactivity Induced by a Perturbation in a Critical Reactor

- Remember the definition of the static reactivity:

$$\rho = 1 - \frac{1}{k}$$

- Hence, we can write the perturbation formula for the reactivity:

$$\rho \cong \frac{\int_V \nu\delta\Sigma_f(\vec{r})\Phi_0^2(\vec{r})dV - \int_V \delta\Sigma_a(\vec{r})\Phi_0^2(\vec{r})dV}{\int_V \nu\Sigma_{f0}(\vec{r})\Phi_0^2(\vec{r})dV}$$

Generalized Definition of Reactivity

- Diffusion equation

$$-D_0 \nabla^2 \Phi(\vec{r}) + \Sigma_a \Phi(\vec{r}) = \frac{1}{k} \nu \Sigma_f \Phi(\vec{r})$$

- It represents the neutron balance

$$\text{Losses} = \frac{1}{k} \text{Productions}$$

- So k can be interpreted as the ratio Productions/Losses

$$k = \frac{\text{Productions}}{\text{Losses}}$$

Generalized Definition of Reactivity

- Reactivity

$$\rho = 1 - \frac{1}{k} = 1 - \frac{1}{\frac{\text{Productions}}{Losses}} = \frac{\text{Productions} - Losses}{\text{Productions}}$$

- The generalized definition can be also applied to non-static (time-dependent) situations

Approximate Solutions of the Point Kinetics Equations

The Prompt Jump Approximation

Consider the following problem:

Plot the time dependence of the neutron population and the precursor population for a reactor with the following parameters (one-delayed-group model):

$$\Lambda = 0.001s$$

$$\beta = 0.005$$

$$\lambda = 0.2s$$

The reactor is initially critical and a control rod a reactivity worth of 2 mk is inserted at time t=0.

Exact Solution:

Point kinetics with one group of delayed neutrons.

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \lambda \, C$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda C(t)$$

Matrix Form of the Point Kinetics Equations

$$\frac{d}{dt}\begin{bmatrix} n(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\rho-\beta}{\Lambda} & \lambda \\ \dfrac{\beta}{\Lambda} & -\lambda \end{bmatrix} \begin{bmatrix} n(t) \\ C(t) \end{bmatrix}$$

To solve it, first we need to find two particular (fundamental) solutions of the type:

$$\begin{bmatrix} n^{1,2}(t) \\ C^{1,2}(t) \end{bmatrix} = \begin{bmatrix} n_0^{1,2} \\ C_0^{1,2} \end{bmatrix} e^{\omega_{1,2} t}$$

We then write the general solution as a linear combination of those two solutions.

$$\begin{bmatrix} n(t) \\ C(t) \end{bmatrix} = a_1 \begin{bmatrix} n^1(t) \\ C^1(t) \end{bmatrix} + a_2 \begin{bmatrix} n^2(t) \\ C^2(t) \end{bmatrix} = a_1 \begin{bmatrix} n_0^1 \\ C_0^1 \end{bmatrix} e^{\omega_1 t} + a_2 \begin{bmatrix} n_0^2 \\ C_0^2 \end{bmatrix} e^{\omega_2 t}$$

To find the fundamental solutions we substitute their form into the matrix form of the equation, to get:

$$\frac{d}{dt}\left(\begin{bmatrix} n_0 \\ C_0 \end{bmatrix} e^{\omega t} \right) = \begin{bmatrix} \frac{\rho - \beta}{\Lambda} & \lambda \\ \frac{\beta}{\Lambda} & -\lambda \end{bmatrix} \begin{bmatrix} n_0 \\ C_0 \end{bmatrix} e^{\omega t}$$

and, subsequently

$$\omega \begin{bmatrix} n_0 \\ C_0 \end{bmatrix} e^{\omega t} = \begin{bmatrix} \dfrac{\rho - \beta}{\Lambda} & \lambda \\ \dfrac{\beta}{\Lambda} & -\lambda \end{bmatrix} \begin{bmatrix} n_0 \\ C_0 \end{bmatrix} e^{\omega t}$$

$$\omega \begin{bmatrix} n_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} \dfrac{\rho - \beta}{\Lambda} & \lambda \\ \dfrac{\beta}{\Lambda} & -\lambda \end{bmatrix} \begin{bmatrix} n_0 \\ C_0 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\rho-\beta}{\Lambda}-\omega & \lambda \\ \dfrac{\beta}{\Lambda} & -\lambda-\omega \end{bmatrix} \begin{bmatrix} n_0 \\ C_0 \end{bmatrix} = 0$$

The above is a homogeneous linear system:

$$\begin{cases} \left(\dfrac{\rho-\beta}{\Lambda}-\omega\right) n_0 + \lambda C_0 = 0 \\ \dfrac{\beta}{\Lambda} n_0 - (\lambda+\omega) C_0 = 0 \end{cases}$$

We then isolate each unknown on each side of the equations

$$\begin{cases} \left(\dfrac{\rho - \beta}{\Lambda} - \omega \right) n_0 = -\lambda C_0 \\ \dfrac{\beta}{\Lambda} n_0 = (\lambda + \omega) C_0 \end{cases}$$

We now divide the two equations side by side:

$$\frac{\left(\dfrac{\rho - \beta}{\Lambda} - \omega \right)}{\dfrac{\beta}{\Lambda}} = -\frac{\lambda}{(\lambda + \omega)}$$

Which is equivalent to:

$$\left(\frac{\rho-\beta}{\Lambda}-\omega\right)(\lambda+\omega)=-\lambda\frac{\beta}{\Lambda}$$

The above is just a quadratic equation in omega:

$$-\omega^2+\omega\left(\frac{\rho-\beta}{\Lambda}-\lambda\right)+\lambda\frac{\rho-\beta}{\Lambda}+\lambda\frac{\beta}{\Lambda}=0$$

The equation further simplifies to:

$$-\omega^2 + \omega\left(\frac{\rho - \beta}{\Lambda} - \lambda\right) + \lambda\frac{\rho}{\Lambda} = 0$$

or, by changing all signs:

$$\omega^2 - \omega\left(\frac{\rho - \beta}{\Lambda} - \lambda\right) - \lambda\frac{\rho}{\Lambda} = 0$$

This a quadratic equation of the type:

$$a\omega^2 + b\omega + c = 0$$

Solutions of the quadratic equation:

$$\omega_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the coefficients for our equation, we obtain:

$$\omega_{1,2} = \frac{\left(\frac{\rho - \beta}{\Lambda} - \lambda\right) \pm \sqrt{\left(\frac{\rho - \beta}{\Lambda} - \lambda\right)^2 + 4\lambda \frac{\rho}{\Lambda}}}{2}$$

By inserting the numerical values we obtain:

$$\omega_1 = \frac{\left(\frac{-0.002-0.005}{0.001}-0.2\right)+\sqrt{\left(\frac{-0.002-0.005}{0.001}-0.2\right)^2-4\times 0.2\frac{0.002}{0.001}}}{2} =$$

$$= \frac{-7.2+7.088018}{2} = -0.055991[s^{-1}]$$

$$\omega_2 = \frac{\left(\frac{-0.002-0.005}{0.001}-0.2\right)-\sqrt{\left(\frac{-0.002-0.005}{0.001}-0.2\right)^2-4\times 0.2\frac{0.002}{0.001}}}{2} =$$

$$= \frac{-7.2-7.088018}{2} = -7.144009[s^{-1}]$$

We can now use either the first or the second equation to find C_0 as a function of n_0. If we use the second equation, we have:

$$\frac{\beta}{\Lambda} n_0 = (\lambda + \omega_1) C_0$$

Which yields:

$$C_0^1 = \frac{\frac{\beta}{\Lambda}}{(\lambda + \omega_1)} n_0^1 = \frac{\frac{0.005}{0.001}}{0.2 - 0.055991} n_0^1 = 34.720052 n_0^1$$

We can choose n_0^1 to have any value, and for simplicity we will make it unity. So our first fundamental solution will be:

$$\begin{bmatrix} 1 \\ 34.720052 \end{bmatrix} e^{-0.055991\times t}$$

To obtain the second fundamental solution we use the second equation, and substitute the value of the second eigenvalue, ω_2.

$$C_0^2 = \frac{\frac{\beta}{\Lambda}}{(\lambda + \omega_2)} n_0^2 = \frac{\frac{0.005}{0.001}}{0.2 - 7.144009} n_0^2 = -0.720045 n_0^1$$

Our second fundamental solution is then:

$$\begin{bmatrix} 1 \\ -0.720045 \end{bmatrix} e^{-7.144009\times t}$$

The general solution is then:

$$\begin{bmatrix} n(t) \\ C(t) \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 34.720052 \end{bmatrix} e^{-0.055991\times t} + a_2 \begin{bmatrix} 1 \\ -0.720045 \end{bmatrix} e^{-7.144009\times t}$$

where a_1 and a_2 are constants that need to be found by applying the initial conditions, $\begin{bmatrix} n(0) \\ C(0) \end{bmatrix}$.

We choose t=0 to be the time when the control device is inserted. The initial conditions for the subsequent evolution of the reactor are the steady-state conditions present right before the control device is removed from the core. It means that we need to find n and C for the steady state.

We need to recognize that the point kinetics equations are also valid for steady state operation.

A reactor is said to operate at steady state when its power, neutron population, and precursor population do not change with time. For a reactor to operate at steady state, it has to be critical, that is its reactivity has to be zero.
For a critical reactor, the point kinetics equations with one group of delayed neutrons are written:

$$\frac{dn(t)}{dt} = \frac{-\beta}{\Lambda} n(t) + \lambda\, C$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda C(t)$$

Now, accounting for the steady state, which means that the time derivatives of the neutron population and precursor population are zero, we can re-write the (one delayed-group) point kinetics equations as:

$$0 = \frac{-\beta}{\Lambda} n_{steady} + \lambda\, C_{steady}$$

$$0 = \frac{\beta}{\Lambda} n_{steady} - \lambda C_{steady}$$

We can see right away that, for steady-state conditions, the second equation is equivalent to the first, just taken with a minus sign.

We can now use either of the two point kinetics equations to find :

$$C_{steady} = \frac{\beta}{\lambda \Lambda} n_{steady}$$

$$C_{steady} = \frac{0.005}{0.2 \times 0.001} n_{steady} = 25 \times n_{steady}$$

So now we can impose the initial conditions:

$$\begin{bmatrix} n_{steady} \\ C_{steady} \end{bmatrix} = \begin{bmatrix} n(0) \\ C(0) \end{bmatrix} =$$

$$= a_1 \begin{bmatrix} 1 \\ 34.720052 \end{bmatrix} e^{-0.055991\times 0} + a_2 \begin{bmatrix} 1 \\ -0.720045 \end{bmatrix} e^{-7.144009\times 0} =$$

$$= a_1 \begin{bmatrix} 1 \\ 34.720052 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -0.720045 \end{bmatrix}$$

$$\begin{bmatrix} n_{steady} \\ C_{steady} \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 34.720052 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -0.720045 \end{bmatrix}$$

This yields the following linear system:

$$a_1 + a_2 = n_{steady}$$
$$34.720052 \times a_1 - 0.720045 \times a_2 = 25 \times n_{steady}$$

We solve the system by reduction:

Firstly, we multiply the first equation by an appropriate number

$$0.720045 \times a_1 + 0.720045 \times a_2 = 0.720045 \times n_{steady}$$
$$34.720052 \times a_1 - 0.720045 \times a_2 = 25 \times n_{steady}$$

Secondly, we add the two equations:

$$35.440097 \times a_1 = 25.720045 \times n_{steady}$$

$$a_1 = \frac{25.720045}{35.440097} \times n_{steady} = 0.725733 \times n_{steady}$$

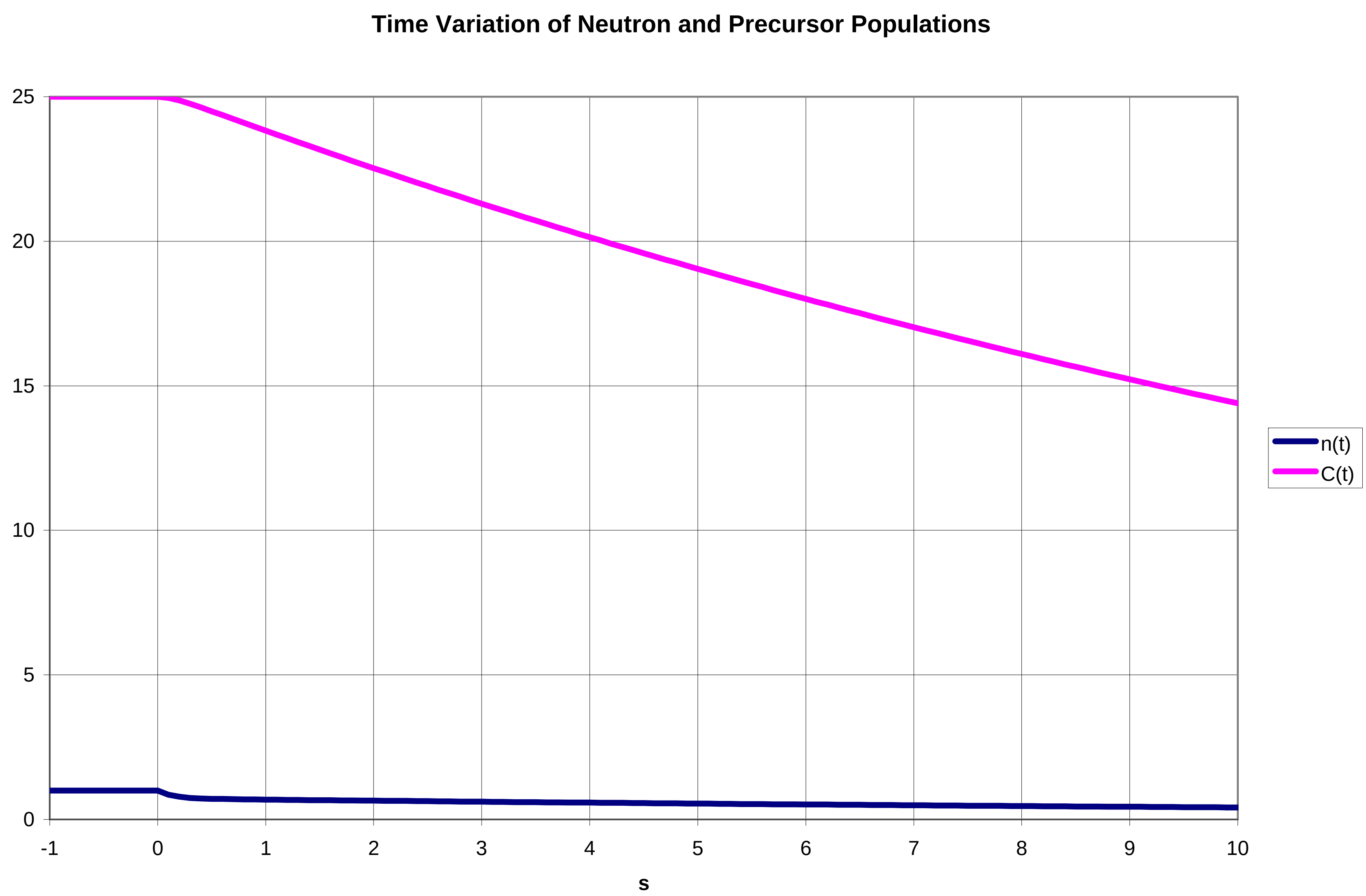
$$a_2 = n_{steady} - a_1 = n_{steady} - 0.725733 \times n_{steady} = 0.274267 \times n_{steady}$$

The solution is hence:

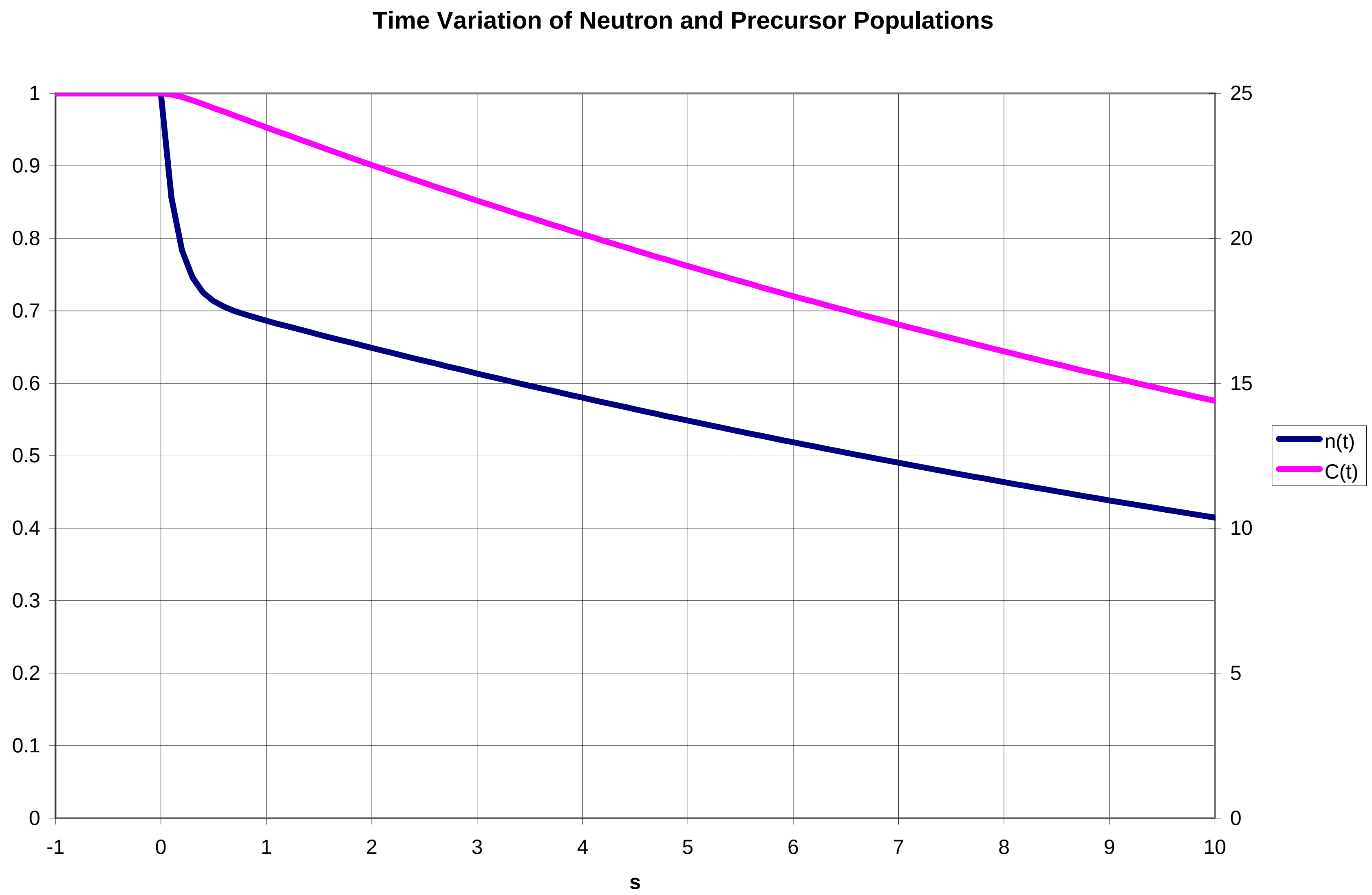
$$\begin{bmatrix} n(t) \\ C(t) \end{bmatrix} = n_{steady} \begin{pmatrix} 0.725733 \times \begin{bmatrix} 1 \\ 34.720052 \end{bmatrix} e^{-0.055991 \times t} + \\ +0.274267 \begin{bmatrix} 1 \\ -0.720045 \end{bmatrix} e^{-7.144009 \times t} \end{pmatrix}$$

$$n(t) = n_{steady} \left(0.725733 \times e^{-0.055991 \times t} + 0.274267 e^{-7.144009 \times t} \right)$$

$$C(t) = n_{steady} \begin{pmatrix} 0.725733 \times 34.720052 e^{-0.055991 \times t} + \\ 0.274267 \times 0.720045 e^{-7.144009 \times t} \end{pmatrix}$$



Time Variation of Neutron and Precursor Populations
25
20
15
10
5
0
-1
0
1
2
3
4
5
6
7
8
9
10
s
n(t)
C(t)



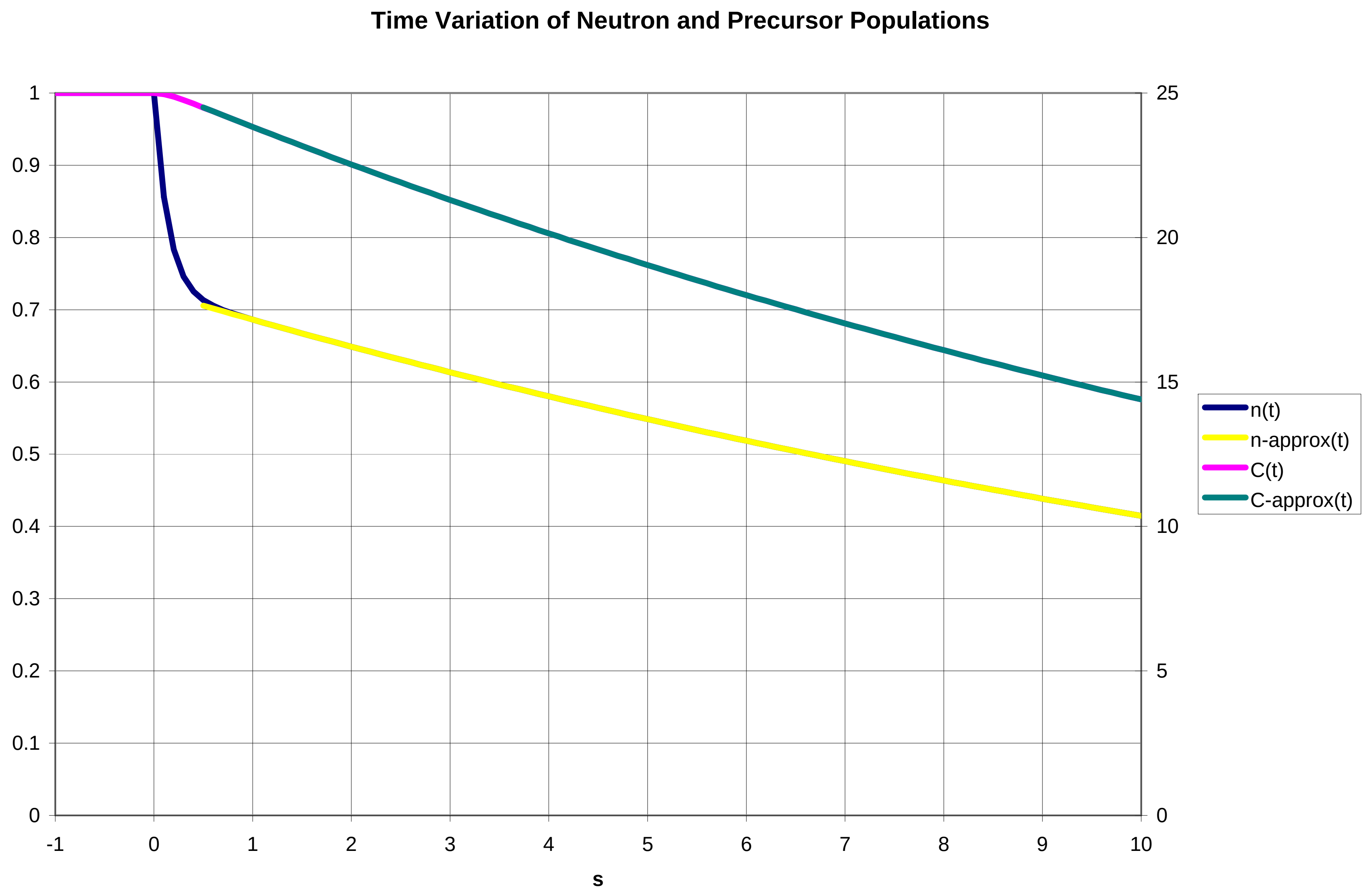
After t=0.5s we can approximate:

$$n(t) = n_{steady}\left(0.725733 \times e^{-0.055991 \times t} + 0.274267 e^{-7.144009 \times t}\right) \cong$$

$$\cong n_{steady} \times 0.725733 \times e^{-0.055991 \times t}$$

$$C(t) = n_{steady}\left(\begin{matrix} 0.725733 \times 34.720052 e^{-0.055991 \times t} + \\ 0.274267 \times 0.720045 e^{-7.144009 \times t} \end{matrix}\right) \cong$$

$$\cong n_{steady} \times 0.725733 \times 34.720052 e^{-0.055991 \times t}$$

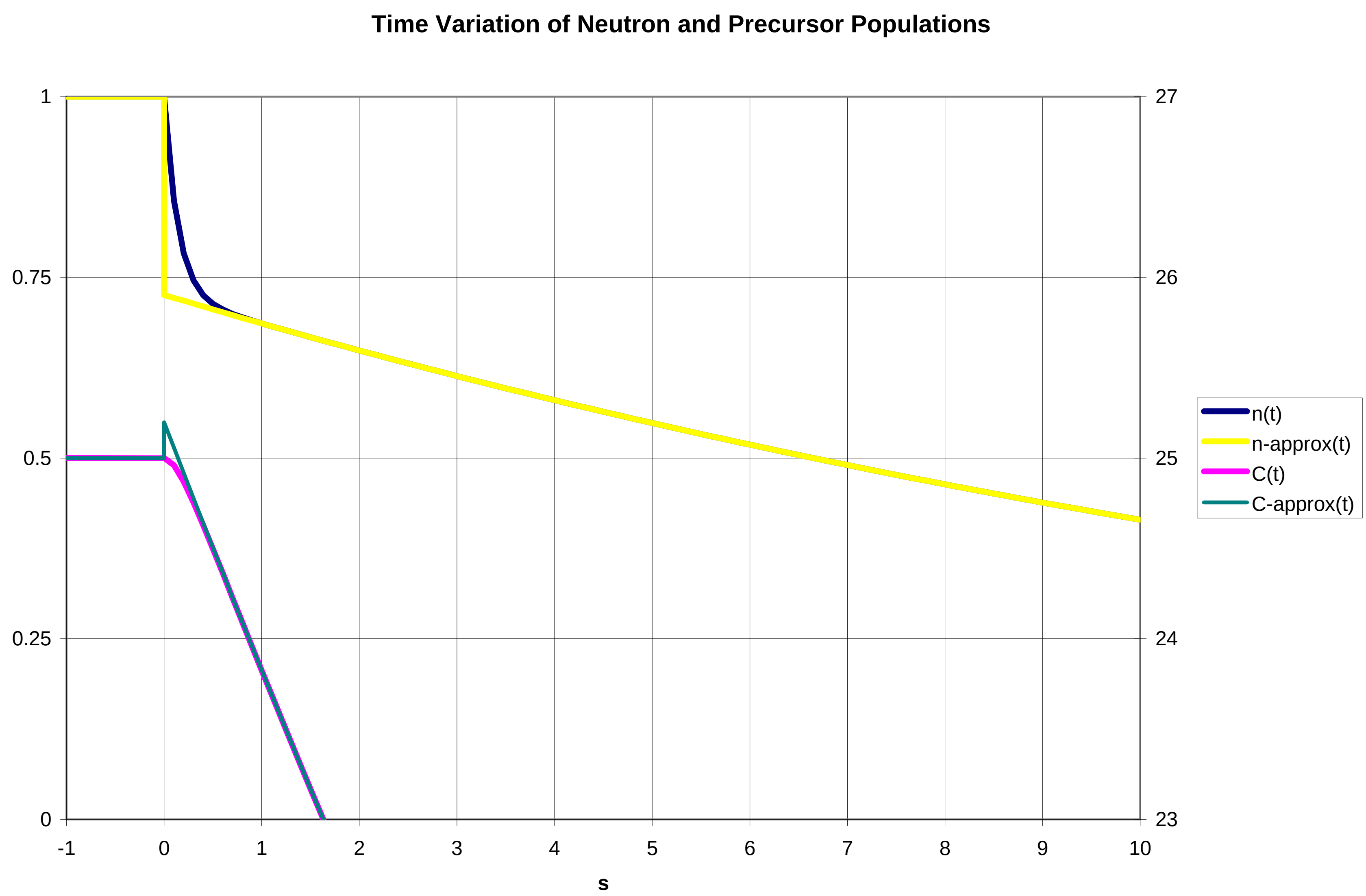


Time Variation of Neutron and Precursor Populations
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
25
20
15
10
5
0
-1
0
1
2
3
4
5
6
7
8
9
10
s
n(t)
n-approx(t)
C(t)
C-approx(t)

In fact, we can use the same approximation up to $t = 0^+$ and assume both n and C have a "prompt jump" at t=0.

$$n(t) \cong \begin{cases} n_{steady} & t \le 0 \\ n_{steady} \times 0.725733 \times e^{-0.055991 \times t} & t > 0 \end{cases}$$

$$C(t) \cong \begin{cases} n_{steady} \times 25 & t \le 0 \\ n_{steady} \times 0.725733 \times 34.720052 e^{-0.055991 \times t} & t > 0 \end{cases}$$

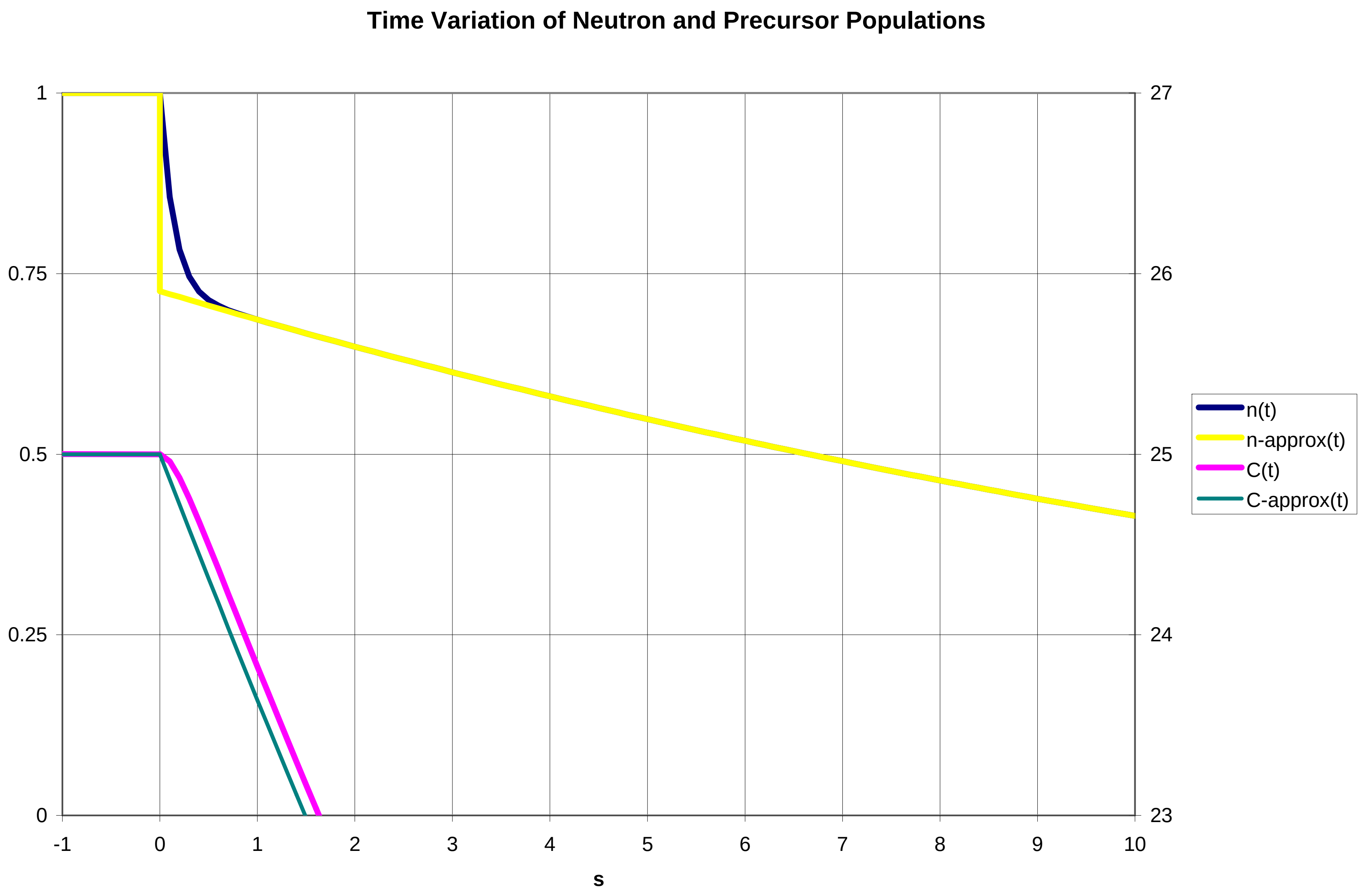


Time Variation of Neutron and Precursor Populations
1
0.75
0.5
0.25
0
27
26
25
24
23
-1
0
1
2
3
4
5
6
7
8
9
10
s
n(t)
n-approx(t)
C(t)
C-approx(t)

Because the jump in the precursor population is small, we can neglect it, and write:

$$n(t) \cong \begin{cases} n_{steady} & t \le 0 \\ n_{steady} \times 0.725733 \times e^{-0.055991 \times t} & t > 0 \end{cases}$$

$$C(t) \cong \begin{cases} n_{steady} \times 25 & t \le 0 \\ n_{steady} \times 25 \times e^{-0.055991 \times t} & t > 0 \end{cases}$$



Time Variation of Neutron and Precursor Populations
1
0.75
0.5
0.25
0
27
26
25
24
23
-1
0
1
2
3
4
5
6
7
8
9
10
s
n(t)
n-approx(t)
C(t)
C-approx(t)

The trick is now to use these approximations beforehand, to come up with simplified point kinetics equations.

On top of the approximations stated so far, we need to make one more observation:

$$|\omega_2| << \left|\frac{\rho - \beta}{\Lambda}\right|$$

Indeed, in our case

$$0.055991 = |\omega_2| << \left|\frac{\rho - \beta}{\Lambda}\right| = \left|\frac{0.002 - 0.005}{0.001}\right| = 3$$

Approximations:

- The neutron population suffers a prompt jump at t=0

$$n(0^+) \neq n(0)$$

- After the jump, the neutron population has a simple exponential behaviour.

$$n(t) = n(0^+)e^{\omega t}$$

- The precursor population does not have a jump at t=0 (it is continuous)

$$C(0^+) = C(0)$$

- The precursor population follows a simple exponential behaviour, with the same exponent as the neutron population.

$$C(t) = C(0^+)e^{\omega t} = C(0)e^{\omega t}$$

- $$|\omega_2| << \left|\frac{\rho - \beta}{\Lambda}\right|$$

With the above approximations, the point kinetics equations with one delayed-neutron groups are written (for $t \le 0$ and critical reactor):

$$0 = \frac{-\beta}{\Lambda} n + \lambda\, C$$

$$0 = \frac{\beta}{\Lambda} n - \lambda C$$

At t=0, we have:

$$0 = \frac{-\beta}{\Lambda} n(0) + \lambda\, C(0)$$

$$0 = \frac{\beta}{\Lambda} n(0) - \lambda C(0)$$

The two equations are equivalent and each provides the relationship between the precursor population and the neutron population at steady state.

$$C(0) = \frac{\beta}{\lambda\Lambda} n(0)$$

For t>0 we have:

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \lambda\ C$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda C(t)$$

We will now simplify the first equation

Substituting the approximate expressions for n and C we obtain:

$$\frac{d}{dt}\left[n(0^+)e^{\omega t}\right] = \frac{\rho-\beta}{\Lambda}n(0^+)e^{\omega t} + \lambda\, C(0)e^{\omega t}$$

Simple processing yields:

$$\omega n(0^+)e^{\omega t} = \frac{\rho-\beta}{\Lambda}n(0^+)e^{\omega t} + \lambda\, C(0)e^{\omega t}$$

Because

$$|\omega| << \left|\frac{\rho - \beta}{\Lambda}\right|$$

We can write:

$$\left|\omega n(0^+)e^{\omega t}\right| << \left|\frac{\rho - \beta}{\Lambda} n(0^+)e^{\omega t}\right|$$

The above relationship allows us to neglect $\left|\omega n(0^+)e^{\omega t}\right|$ in the equation and thus we can write:

$$0 = \frac{\rho - \beta}{\Lambda} n(0^+)e^{\omega t} + \lambda\, C(0)e^{\omega t}$$

Remembering that

$$n(0^+)e^{\omega t} = n(t)$$
$$C(0)e^{\omega t} = C(t)$$

We can re-write the above as:

$$0 = \frac{\rho - \beta}{\Lambda} n(t) + \lambda\, C(t)$$

We have thus eliminated the time derivative in the first equation. The one-delayed-group point kinetics equations in the prompt jump approximation are thus written:

$$0 = \frac{\rho - \beta}{\Lambda} n(t) + \lambda\, C$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda C(t)$$

Note that the first equation is now reduced to a simple algebraic equation instead of a differential equation. Thus our system of two differential equations has been reduced to just one differential equation.

Prompt Jump Approximation

- Reactor is at steady state for t<0
- Neutron population has a jump at t=0
- Precursor population is continuous at t=0
- For t>0, $\frac{dn}{dt} = 0$

NOTE: The same assumptions are applied regardless of the sign of the reactivity

Application of the prompt jump approximation to our initial problem.

Before inserting the rod:

$$0 = \frac{-\beta}{\Lambda} n(0) + \lambda\, C(0)$$

$$0 = \frac{\beta}{\Lambda} n(0) - \lambda C(0)$$

Either equation yields:

$$C(0) = \frac{\beta}{\lambda \Lambda} n(0)$$

Right after the rod insertion:

$$0 = \frac{\rho - \beta}{\Lambda} n(t) + \lambda\, C(t)$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda C(t)$$

The first equation yields directly:

$$n(t) = -\frac{\lambda \Lambda}{\rho - \beta} C(t)$$

Substituting this into the second equation, we obtain:

$$\frac{dC(t)}{dt} = -\frac{\beta}{\Lambda}\frac{\lambda\Lambda}{\rho-\beta}C(t) - \lambda C(t)$$

Simple processing results in:

$$\frac{dC(t)}{dt} = -\left(\frac{\beta\lambda}{\rho-\beta} + \frac{\rho\lambda-\beta\lambda}{\rho-\beta}\right)C(t)$$

$$\frac{dC(t)}{dt} = \frac{\rho\lambda}{\beta-\rho}C(t)$$

The solution of the above is, obviously an exponential:

$$C(t) = C(0)e^{\frac{\rho\lambda}{\beta-\rho}t}$$

The neutron population is given by:

$$n(t) = -\frac{\lambda\Lambda}{\rho-\beta}C(t) = -\frac{\lambda\Lambda}{\rho-\beta}C(0)e^{\frac{\rho\lambda}{\beta-\rho}t}$$

At $t = 0^+$, we have:

$$n(0^+) = -\frac{\lambda\Lambda}{\rho-\beta}C(0)$$

Substituting the value of C(0) from the steady state solution

$$C(0) = \frac{\beta}{\lambda\Lambda} n(0)$$

(we can do that because C is continuous at t=0) we obtain:

$$C(t) = C(0)e^{\frac{\rho\lambda}{\beta-\rho}t} = \frac{\beta}{\lambda\Lambda} n(0)e^{\frac{\rho\lambda}{\beta-\rho}t}$$

$$n(t) = -\frac{\lambda\Lambda}{\rho-\beta} C(0)e^{\frac{\rho\lambda}{\beta-\rho}t} = \frac{\beta}{\beta-\rho} n(0)e^{\frac{\rho\lambda}{\beta-\rho}t}$$

At $t = 0^{+}$ we have:

$$n(0^{+}) = -\frac{\lambda\Lambda}{\rho - \beta} C(0) = \frac{\beta}{\beta - \rho} n(0)$$

And hence a jump in the neutron population which used to be equal to n(0) for t<0.

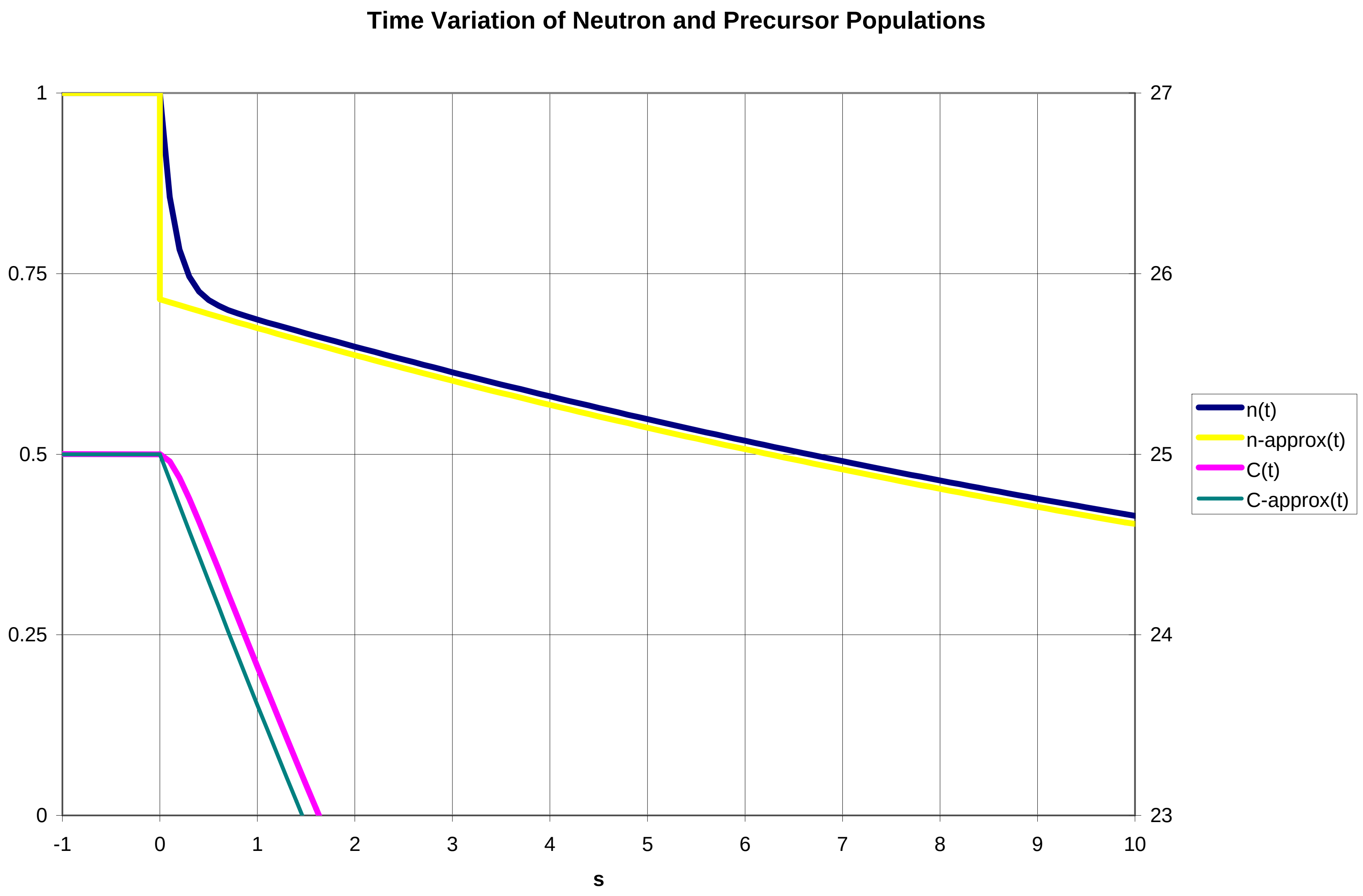
Substituting the numerical values we obtain for t>0:

$$C(t) = \frac{\beta}{\lambda\Lambda} n(0) e^{\frac{\rho\lambda}{\beta-\rho}t} = n(0)\frac{0.005}{0.2\times 0.001} e^{\frac{-0.002\times 0.2}{0.005+0.002}t} =$$

$$= n(0)\times 25 \times e^{-0.057143\times t}$$

$$n(t) = \frac{\beta}{\beta-\rho} n(0) e^{\frac{\rho\lambda}{\beta-\rho}t} = n(0)\frac{0.005}{0.005+0.002} e^{-0.057143\times t} =$$

$$= n(0)\times 0.714286 \times e^{-0.057143\times t}$$



Time Variation of Neutron and Precursor Populations
1
0.75
0.5
0.25
0
27
26
25
24
23
-1
0
1
2
3
4
5
6
7
8
9
10
s
n(t)
n-approx(t)
C(t)
C-approx(t)

The prompt jump approximation for the case of 6 delayed-neutron groups

Inhour equation

Start with the point kinetics equations

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{k=1}^{6} \lambda_k \hat{C}_k$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda} n(t) - \lambda_k \hat{C}_k(t), \quad k = 1...6$$

This is a system of seven coupled differential equations with constant coefficients.

Solutions of the form

$$\begin{bmatrix} ne^{\omega t} \\ c_1 e^{\omega t} \\ \vdots \\ c_6 e^{\omega t} \end{bmatrix} = \begin{bmatrix} n \\ c_1 \\ \vdots \\ c_6 \end{bmatrix} e^{\omega t}$$

Substituting the above form we obtain for each fundamental solution (there are 7 of them):

$$\omega n = \frac{\rho - \beta}{\Lambda} n + \sum_{k=1}^{6} \lambda_k c_k$$

$$\omega c_k = \frac{\beta}{\Lambda} n - \lambda_k c_k, \quad k = 1...6$$

Solving for c_k in the precursor equations, we obtain:

$$c_k = \frac{\beta_k}{\Lambda(\omega + \lambda_k)} n, \quad k = 1...6$$

Substituting the above back into the first (neutron balance) equation, we obtain the inhour equation:

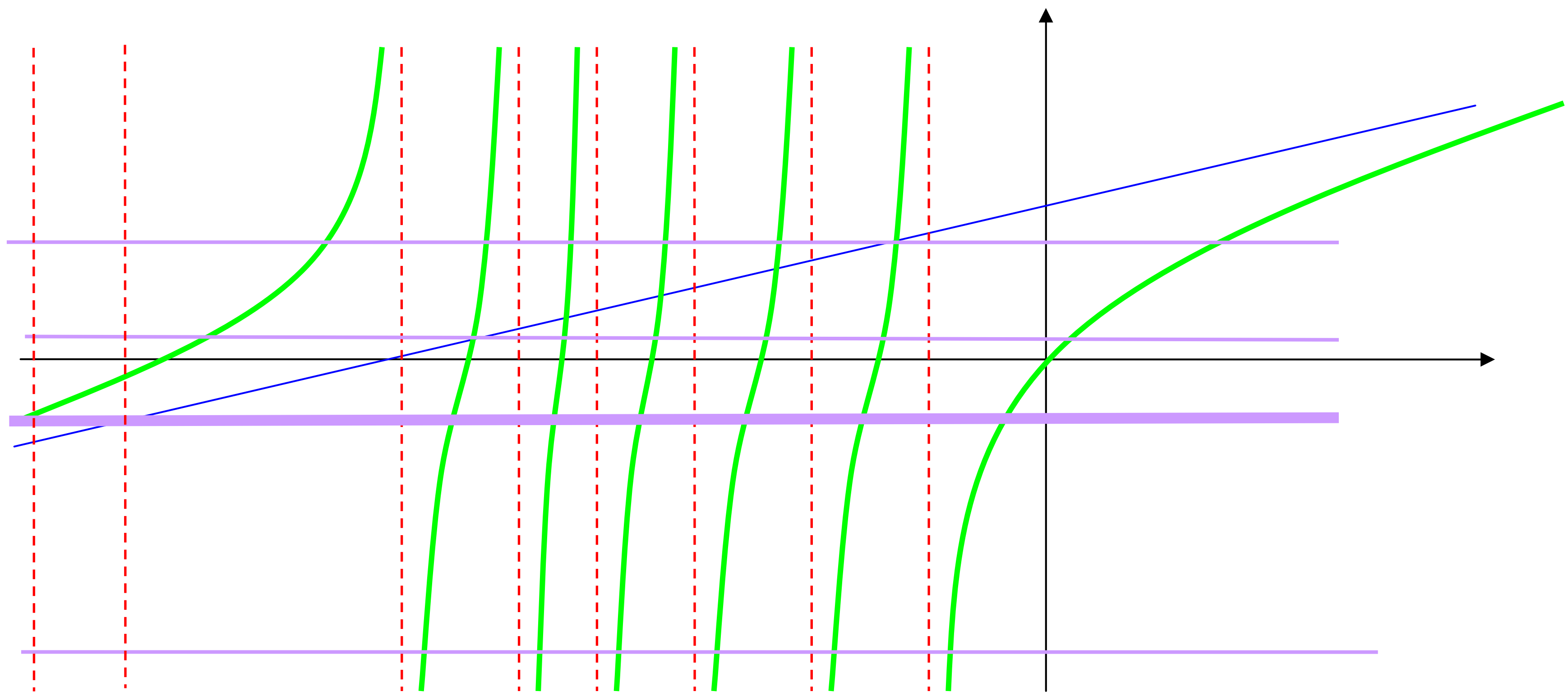
$$\rho = \Lambda\omega + \beta - \sum_{k=1}^{6} \lambda_k \frac{\beta_k}{(\omega + \lambda_k)}$$

We can solve graphically for ω by plotting the RHS and intersecting it with a horizontal line at y=ρ.

For large (positive or negative) values of ω we obtain the asymptotic behaviour:

$$\rho = \Lambda\omega + \beta - \sum_{k} \lambda_k \frac{\beta_k}{\infty + \lambda_k} = \Lambda\omega + \beta$$

Note that : $0.0001 < \Lambda < 0.001$, so the slope is very small



Reactor Period

$$T = \frac{1}{\omega_{\max}}$$

Because of the very small slope, there is one omega that is much smaller than all the others:

$$\omega_7 << \omega_6 < \omega_5 < \omega_4 < \omega_3 < \omega_2 < \omega_1$$

$$\omega_7 << \lambda_6 < \lambda_5 < \lambda_4 < \lambda_3 < \lambda_2 < \lambda_1$$

In absolute value, it is much larger than all the rest, of course.

Also because of the small slope of the asymptote, its intersection point with the horizontal line defined by ρ is also far to the left, so:

$$\frac{\rho - \beta}{\Lambda} << \omega_6 < \omega_5 < \omega_4 < \omega_3 < \omega_2 < \omega_1$$

$$\frac{\rho - \beta}{\Lambda} << \lambda_6 < \lambda_5 < \lambda_4 < \lambda_3 < \lambda_2 < \lambda_1$$

Approximating the graph with its asymptote for $\omega \cong \omega_7$ (which is a fairly crude approximation) we obtain:

$$\rho \cong \Lambda\omega_7 + \beta \Rightarrow \omega_7 \cong \frac{\rho - \beta}{\Lambda}$$

Which should only be used to give us an idea of the order of magnitude of ω_7, not for any precise calculations.

The fundamental solution for ω_7 is:

$$\begin{bmatrix} n^7 \\ c_1^7 \\ \vdots \\ c_6^7 \end{bmatrix} e^{\omega_7 t}$$

$$c_k^7 = \frac{\beta_k}{\Lambda\left(\omega_7 + \lambda_7\right)} n^7$$

$\omega_7 + \lambda_7 < 0$ hence $c_k^7 < 0$

For $\omega_{i\neq 7}$:

$$\begin{bmatrix} n^i \\ c_1 \\ \vdots \\ c_6^i \end{bmatrix} e^{\omega_i t}$$

It follows that for $\omega_{i\neq 7}$ (i.e. after the fastest exponential has died out):

$$\omega_i n^i = \frac{\rho - \beta}{\Lambda} n^i + \sum_{k=1}^{6} \lambda_k c_k^i$$

Since:

$$\left|\omega_{i\neq 7}\right| << \left|\frac{\rho-\beta}{\Lambda}\right|$$

we can write:

$$0 \cong \frac{\rho-\beta}{\Lambda} n^i + \sum_{k=1}^{6} \lambda_k c_k^i$$

Because the general solution is just a linear combination of the fundamental solutions, the above equation can be extended to the general solution (for times large-enough that the first exponential has died out).

$$0 \cong \frac{\rho-\beta}{\Lambda} n + \sum_{k=1}^{6} \lambda_k \hat{C}_k$$

Prompt Jump Approximation for 6 Delayed-Neutron Groups

$$\left.\begin{aligned} 0 &= \frac{\rho-\beta}{\Lambda} n(t) + \sum_{k=1}^{6} \lambda_k \hat{C}_k \\ \frac{d\hat{C}_k(t)}{dt} &= \frac{\beta_k}{\Lambda} n(t) - \lambda_k \hat{C}_k(t), \quad k = 1 \ldots 6 \end{aligned}\right\} t > 0$$

<u>Assumptions</u>

n suffers a sudden (prompt) jump at t=0;

$\hat{C}_k$ are continuous at t=0.

Reactivity Measurement Methods

Reactivity Measurement Methods

1. Static (Do not involve transients)
   1.1. Source Multiplication Method
   1.2. Null Reactivity Method
2. Dynamic (Involve transients)
   2.1. Asymptotic Period Method
   2.2. Rod Drop Method
   2.3. Source/Rod Jerk Method
   2.4. Pulsed Source Method

Static Reactivity Measurement Methods

Measuring the Reactivity Worth of a Control Rod by the Source Multiplication Method

- The method is applicable for a subcritical reactor with an external neutron source.
- We need to have a second rod whose reactivity worth we already know.

Steps

1. Measure the flux at any reactor position without any of the rods inserted and record the value.
2. Insert the rod of known reactivity worth $\Delta\rho_1$.
3. Measure the flux at the same reactor position as in step 1 and record the value.
4. Remove the rod of known reactivity.
5. Insert the rod of unknown reactivity $\Delta\rho_2$.
6. Measure the flux at the same reactor position as in step 1 and record the value.
7. Calculate the unknown reactivity worth $\Delta\rho_2$.

Calculations

For a subcritical (steady-state) reactor with an external neutron source S, we have:

$$0 = \frac{\rho - \beta}{\Lambda} n + \sum_{k=1}^{6} \lambda_k C_k + S$$

$$0 = \frac{\beta_k}{\Lambda} n - \lambda_k C_k, \quad k = 1 \ldots 6$$

In the above, S is the volume-integrated external source throughout the reactor, and C_k are the precursor populations.

The second equation yields:

$$\frac{\beta_k}{\Lambda} n = \lambda_k C_k, \quad k = 1...6$$

By adding the equations for all six delayed groups we obtain:

$$\frac{\sum_{k=1}^{6} \beta_k}{\Lambda} n = \sum_{k=1}^{6} \lambda_k C_k \Leftrightarrow \frac{\beta}{\Lambda} n = \sum_{k=1}^{6} \lambda_k C_k$$

$$0 = \frac{\rho - \beta}{\Lambda} n + \sum_{k=1}^{6} \lambda_k C_k + S = \frac{\rho - \beta}{\Lambda} n + \frac{\beta}{\Lambda} n + S = \frac{\rho}{\Lambda} n + S$$

It follows that:

$$n = -\frac{\Lambda}{\rho} S$$

(Source Multiplication)

Since

$\hat{\Phi} = n\overline{\mathrm{v}}$

We can write:

$$\hat{\Phi} = -\frac{\Lambda}{\rho}\overline{\mathrm{v}}S$$

If we measure the flux with a detector at a certain position, we can write:

$$\Phi_{detector} = \hat{\Phi}\frac{\Phi_{detector}}{\hat{\Phi}} = -\frac{\Lambda}{\rho}\overline{\nu}\frac{\Phi_{detector}}{\hat{\Phi}}S$$

By denoting:

$$\tilde{S} = \overline{\nu}\frac{\Phi_{detector}}{\hat{\Phi}}S$$

We can write:

$$\Phi_{detector} = -\frac{\Lambda}{\rho}\tilde{S}$$

Which is just another form of the source multiplication formula.

Hence, for any point in a subcritical reactor with an external neutron source we have:

$$\Phi_{\det ector} = -\frac{\Lambda}{\rho}\tilde{S}$$

For the reactor without any control rod we can write:

$$\Phi^0_{\det ector} = -\frac{\Lambda}{\rho_0}\tilde{S} \quad (1)$$

or

$$\frac{1}{\Phi^0_{\det ector}} = -\frac{\tilde{S}}{\Lambda}\rho_0 \quad (2)$$

After we insert the rod of known reactivity, the reactivity becomes

$$\rho_1 = \rho_0 + \Delta\rho_1 \tag{3}$$

The flux equation becomes

$$\frac{1}{\Phi^1_{\det ector}} = -\frac{1}{\Lambda}\frac{1}{\tilde{S}}\left(\rho_0 + \Delta\rho_1\right) \tag{4}$$

Dividing equation (4) by equation (2) we obtain:

$$\frac{\Phi^0_{\det ector}}{\Phi^1_{\det ector}} = \frac{\rho_0 + \Delta\rho_1}{\rho_0} \tag{5}$$

We can now solve for ρ_0

$$\rho_0 = \frac{\Delta\rho_1}{\dfrac{\Phi^0_{\det ector}}{\Phi^1_{\det ector}} - 1} \tag{6}$$

After we remove the rod of known worth and insert the rod of unknown worth, we have:

$$\frac{1}{\Phi^2_{\det ector}} = -\frac{1}{\Lambda}\frac{1}{\tilde{S}}\left(\rho_0 + \Delta\rho_2\right) \tag{7}$$

Dividing now equation (7) by equation (2) we obtain

$$\frac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} = \frac{\rho_0 + \Delta\rho_2}{\rho_0} \tag{8}$$

We can now solve for $\Delta\rho_2$ in (8)

$$\Delta\rho_2 = \rho_0\left(\frac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} - 1\right) \tag{9}$$

And substituting the expression for ρ_0 that we found in equation (6) we obtain:

$$\Delta\rho_2 = \rho_0\left(\frac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} - 1\right) = \Delta\rho_1 \frac{\dfrac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} - 1}{\dfrac{\Phi^0_{\det ector}}{\Phi^1_{\det ector}} - 1} \tag{10}$$

So the formula for finding the reactivity worth for our rod is:

$$\Delta\rho_2 = \Delta\rho_1 \frac{\dfrac{\Phi^0_{\det ector}}{\Phi^2_{\det ector}} - 1}{\dfrac{\Phi^0_{\det ector}}{\Phi^1_{\det ector}} - 1} \qquad (11)$$

Example

A subcritical cylindrical reactor has radius 3m and height 6m. A flux detector is placed at r=1m in the midplane of the reactor, and a neutron source of unknown strength is placed in a position diametrically opposed to the detector. The detector reads initially 1000 units. A control rod worth 1 mk is then inserted into the reactor and the detector reading drops to 500 units. The rod is then removed and another rod of unknown worth is inserted. The detector now reads 250 units. What is the reactivity worth of the new rod?

Answer

Applying eq. (11) we have:

$$\Delta\rho_2 = \Delta\rho_1 \frac{\dfrac{\Phi^0_{detector}}{\Phi^2_{detector}} - 1}{\dfrac{\Phi^0_{detector}}{\Phi^1_{detector}} - 1} = 0.001 \times \frac{\dfrac{1000}{250} - 1}{\dfrac{1000}{500} - 1} = 0.001 \times \frac{4-1}{2-1} = 0.001 \times \frac{3}{1} = 0.003$$

Measuring the Reactivity Worth of a Control Rod by the Null Reactivity Method

Method 1 Steps

1. Insert a calibrated rod fully (up to the maximum depth, d_{max}) into the reactor. (A calibrated rod is a rod for which we know what the reactivity worth is as a function of the depth of insertion $\Delta\rho_c(d)$)
2. Make the reactor critical by modifying other parameters, possibly extracting some poison.
3. Insert the rod to be measured
4. Make the reactor critical again by partially extracting the calibrated rod, up to depth d.
5. Calculate the reactivity worth of the second rod, $\Delta\rho_x$.

Calculations

Consider the reactor with the calibrated control rod extracted, but with all other parameters having the same value as when the calibrated rod was inserted.

Its reactivity would be ρ_0 (unknown)

After the insertion of the calibrated rod, we know that the reactivity is zero.

$$\rho_c = \rho_0 + \Delta\rho_c(d_{\max}) = 0$$

After we insert the second rod and withdraw partially the calibrated rod, the reactor is still critical.

$$\rho_x = \rho_0 + \Delta\rho_c(d_x) + \Delta\rho_x = 0$$

Subtracting these two equations we obtain:

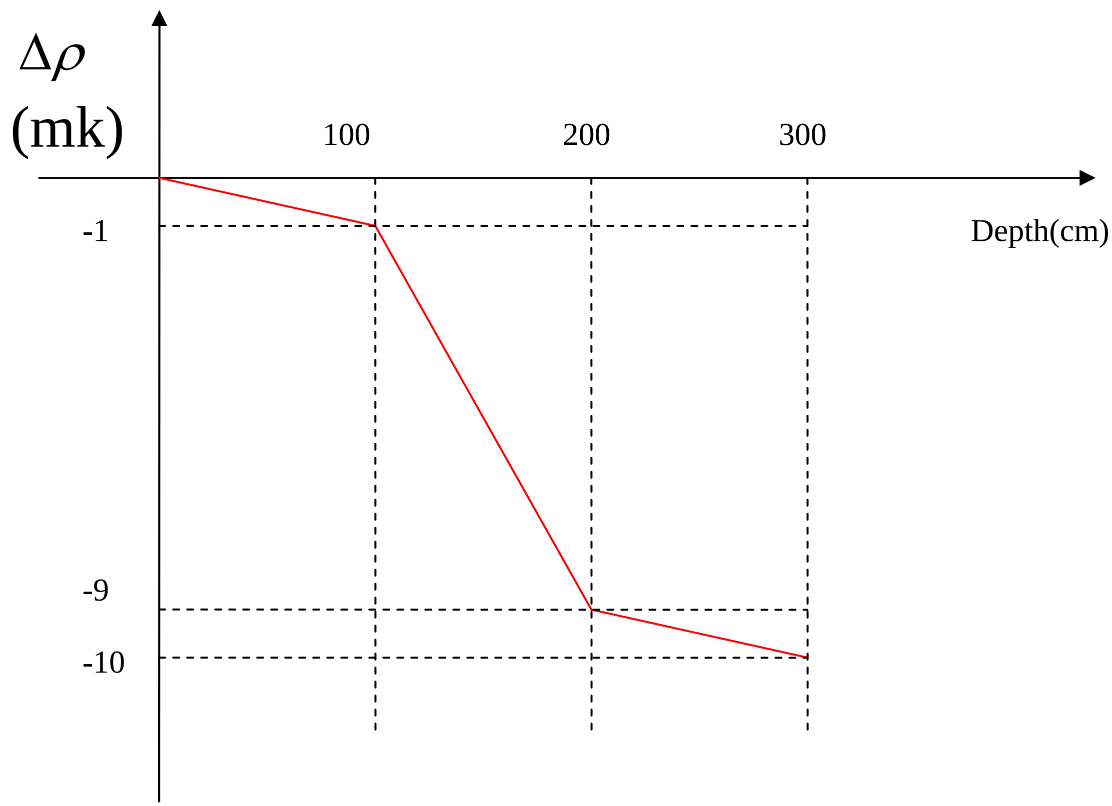
$$\Delta\rho_c(d_x) - \Delta\rho_c(d_{\max}) + \Delta\rho_x = 0$$

We can now solve for $\Delta\rho_x$

$$\Delta\rho_x = \Delta\rho_c(d_{\max}) - \Delta\rho_c(d_x)$$

Example

The reactivity worth of a calibrated control rod as a function of its depth of insertion is given in the graph below. The reactor is made critical with the calibrated rod fully inserted. A second control rod is inserted, and the reactor is again made critical by withdrawing the calibrated control rod up to a depth of 1.5 m. What is the reactivity worth $\Delta\rho_x$ of the second control rod?



$\Delta\rho$
(mk)
100
200
300
-1
Depth(cm)
-9
-10

Solution

Consider the reactor with the calibrated control rod extracted, but with all other parameters having the same value as when the calibrated rod was inserted.

Its reactivity would be ρ_0 (unknown)

After the insertion of the calibrated rod, we know that the reactivity is zero.

$$\rho_c = \rho_0 + \Delta\rho_c(300cm) = 0$$

After we insert the second rod and withdraw partially the calibrated rod, the reactor is still critical.

$$\rho_x = \rho_0 + \Delta\rho_c(150cm) + \Delta\rho_x = 0$$

Subtracting these two equations we obtain:

$$\Delta\rho_c(150cm) - \Delta\rho_c(300cm) + \Delta\rho_x = 0$$

We can now solve for $\Delta\rho_x$

$$\Delta\rho_x = \Delta\rho_c(300cm) - \Delta\rho_c(150cm)$$

Substituting the numerical values, we have:

$$\Delta\rho_x = -10 - (-5) = -5mk$$

Measuring the Reactivity Worth of a Control Rod by the Null Reactivity Method

Method 2 Steps

1. Obtain the reactivity calibration of a liquid poison (e.g. B) as a function of its concentration $\Delta\rho_{poison}(X)$
2. Add some poison while maintaining the reactor critical (possibly by removing some other reactivity devices).
3. Mark the poison concentration Xc.
4. Insert the rod to be measured.
5. Make the reactor critical again by removing some poison.
6. Mark the new poison concentration.
7. Calculate the reactivity worth of the rod, $\Delta\rho_x$.

Calculations

The poison reactivity worth is proportional to the poison concentration:

$$\Delta\rho_{poison}(X) = \alpha X$$

Knowing the reactivity calibration of poison means knowing α , which is usually measured in mk/ppm.

Consider the reactor in step 2 but without any poison. Its reactivity would be ρ_0 (unknown)

Now consider the (critical) reactor at step 2.

$$\rho_c = \rho_0 + \Delta\rho_{poison}(X_c) = 0$$

After we insert the rod and remove part of the poison the reactor is still critical.

$$\rho_x = \rho_0 + \Delta\rho_{poison}(X) + \Delta\rho_x = 0$$

Subtracting these two equations we obtain:

$$\Delta\rho_{poison}(X) - \Delta\rho_{poison}(X_c) + \Delta\rho_x = 0$$

We can now solve for the reactivity worth of the rod.

$$\Delta\rho_x = \Delta\rho_{poison}(X_c) - \Delta\rho_{poison}(X) = \alpha\left(X_c - X\right)$$

Example

The reactivity worth of Boron in a CANDU reactor is 7mk/ppm. The reactor is made critical by the addition of Boron. A control rod is then inserted and the reactor is maintained critical by removing 1.5 ppm of Boron. What is the reactivity worth of the rod?

Solution

By applying the formula we derived, we have:

$$\Delta\rho_x = \alpha\left(X_c - X\right) = 7\times\left(-1.5\right) = -10.5(mk)$$

Reactivity Coefficients

Reactivity Effects

The macroscopic cross sections can change as a consequence of different parameters and, in turn, induce a change in K-eff and hence in reactivity.

The usual parameters that influence the reactivity are:

- Fuel Temperature
- Coolant Temperature
- Moderator Temperature
- Coolant Density

Definition of Reactivity Coefficients

- Consider we keep all the reactor parameters constant, with the exception of one, say the fuel temperature.
- This is not always possible, as a variation in fuel temperature will induce a variation in coolant temperature, but let us assume we can do it.
- Consider we plot the reactivity as a function of the varying parameter (in our case, the fuel temperature).
  $\rho(T_f)$
- We can also plot the reactivity change $\Delta\rho(T_f) = \rho(T_f) - \rho(T_{f0})$ where T_{f0} is the reference fuel temperature.
- This is called the ***reactivity effect*** of fuel temperature.
- We can also calculate and plot $\alpha_{T_f} = \dfrac{d\rho(T_f)}{dT_f}$
- This is called the ***reactivity coefficient*** of the fuel temperature.

Reactivity Effects and Coefficients - Important Facts

It is important to keep in mind what we vary (one or several parameters) and what we keep constant

- For example:
  - Vary the moderator temperature while keeping its density constant
  - Vary the moderator temperature and density to correspond to the temperature.
- Other example:
  - Power reactivity effect and coefficient
  - In this case all changes that stem from the power increase are accounted for (fuel temperature, coolant temperature, coolant density, moderator temperature, moderator density).

Mathematical Expressions of Reactivity Coefficients

Let p be the parameter that is being varied, all others being kept constant.

We define the reactivity coefficient of parameter P as:

$$\alpha_P = \frac{d\rho(p)}{dp}$$

Equivalent definition:

$$\alpha_P = \frac{d\rho(p)}{dp} = \frac{d}{dp}\left(1 - \frac{1}{k(p)}\right) = \frac{1}{k^2(p)}\frac{dk(p)}{dp}$$

Approximate Expressions of Reactivity Coefficients for Near-Critical Reactors

For $k \cong 1$

$$\alpha_P = \frac{1}{k^2(p)}\frac{dk(p)}{dp} \cong \frac{1}{k(p)}\frac{dk(p)}{dp}$$

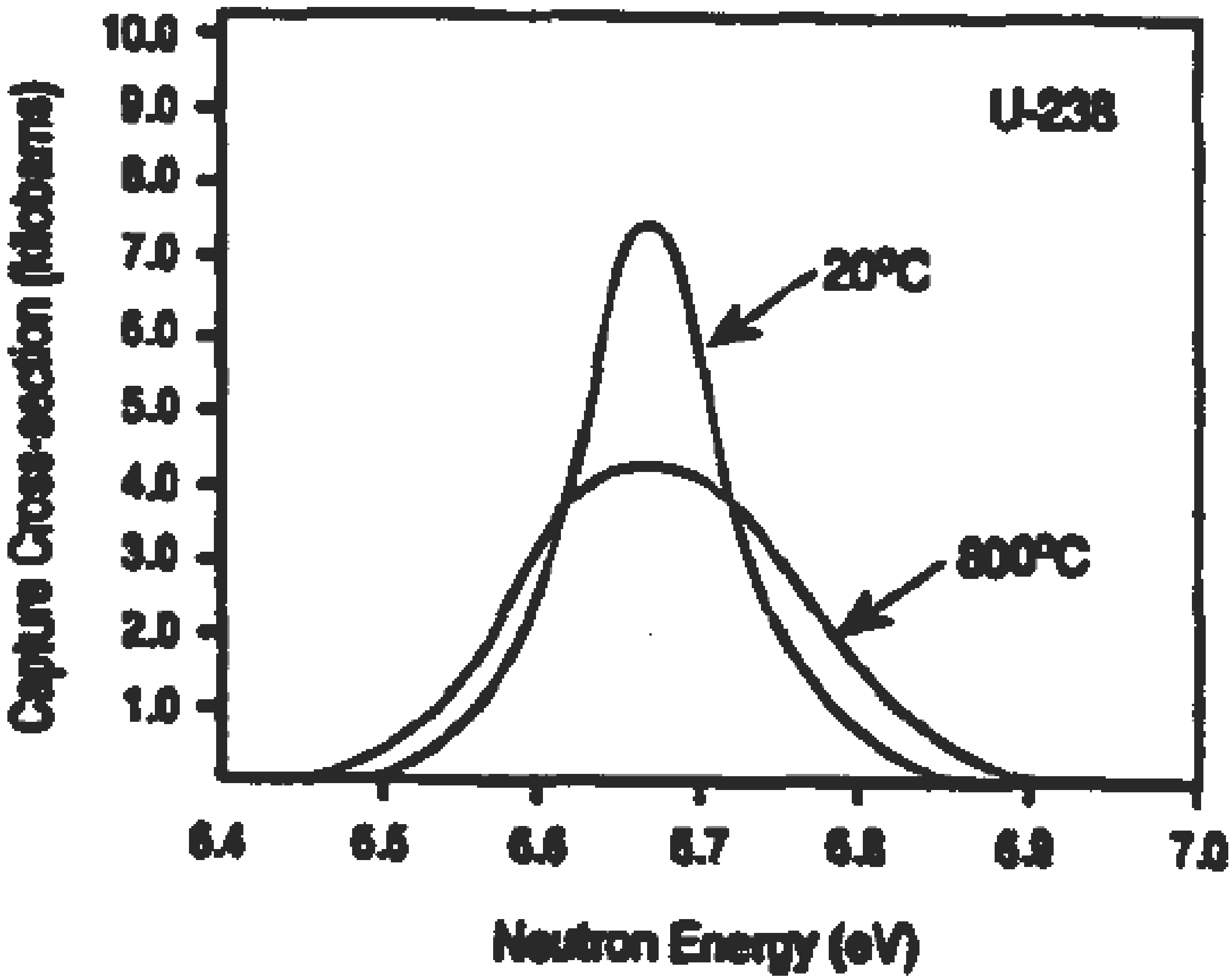
The last form can also be expressed as:

$$\frac{1}{k(p)}\frac{dk(p)}{dp} = \frac{d}{dp}\ln[k(p)]$$

Fuel Temperature Reactivity Effect and Coefficient

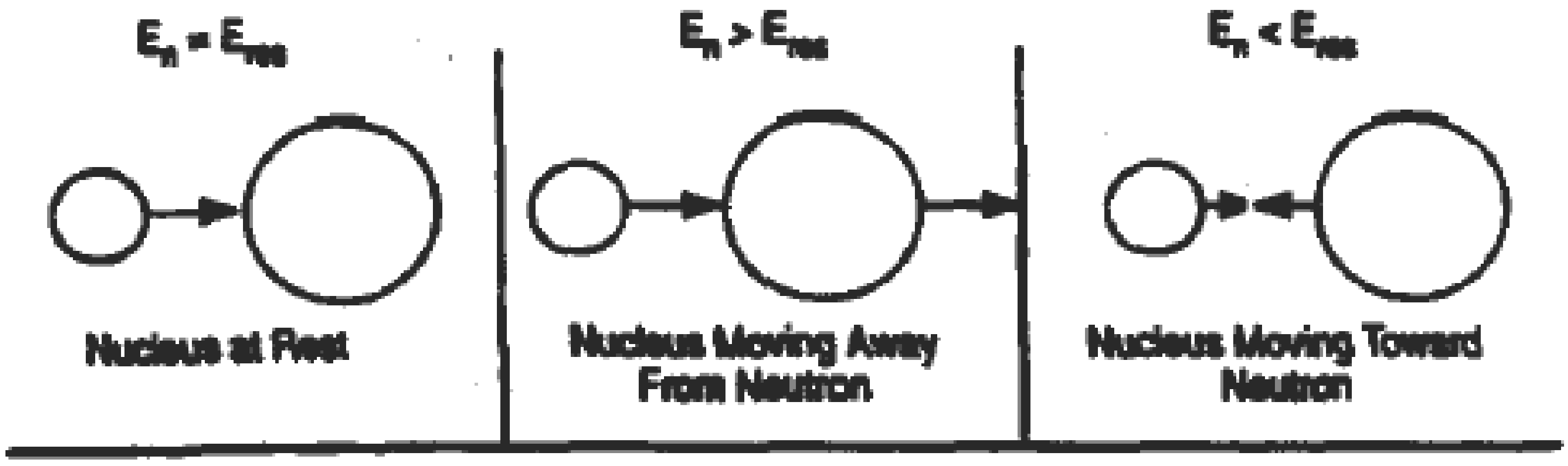
- Assume only the fuel temperature changes
- Due mostly to Doppler broadening of absorption *resonances* of ^{238}U. (A *resonance* is a sharp peak in the energy dependence of the microscopic cross section at a certain energy E_{res}.)
- Reactivity decreases with increasing fuel temperature
- Good safety feature. If reactor power increases accidentally, the reactivity decreases bringing the power back down.

Doppler Effect for Neutron Capture

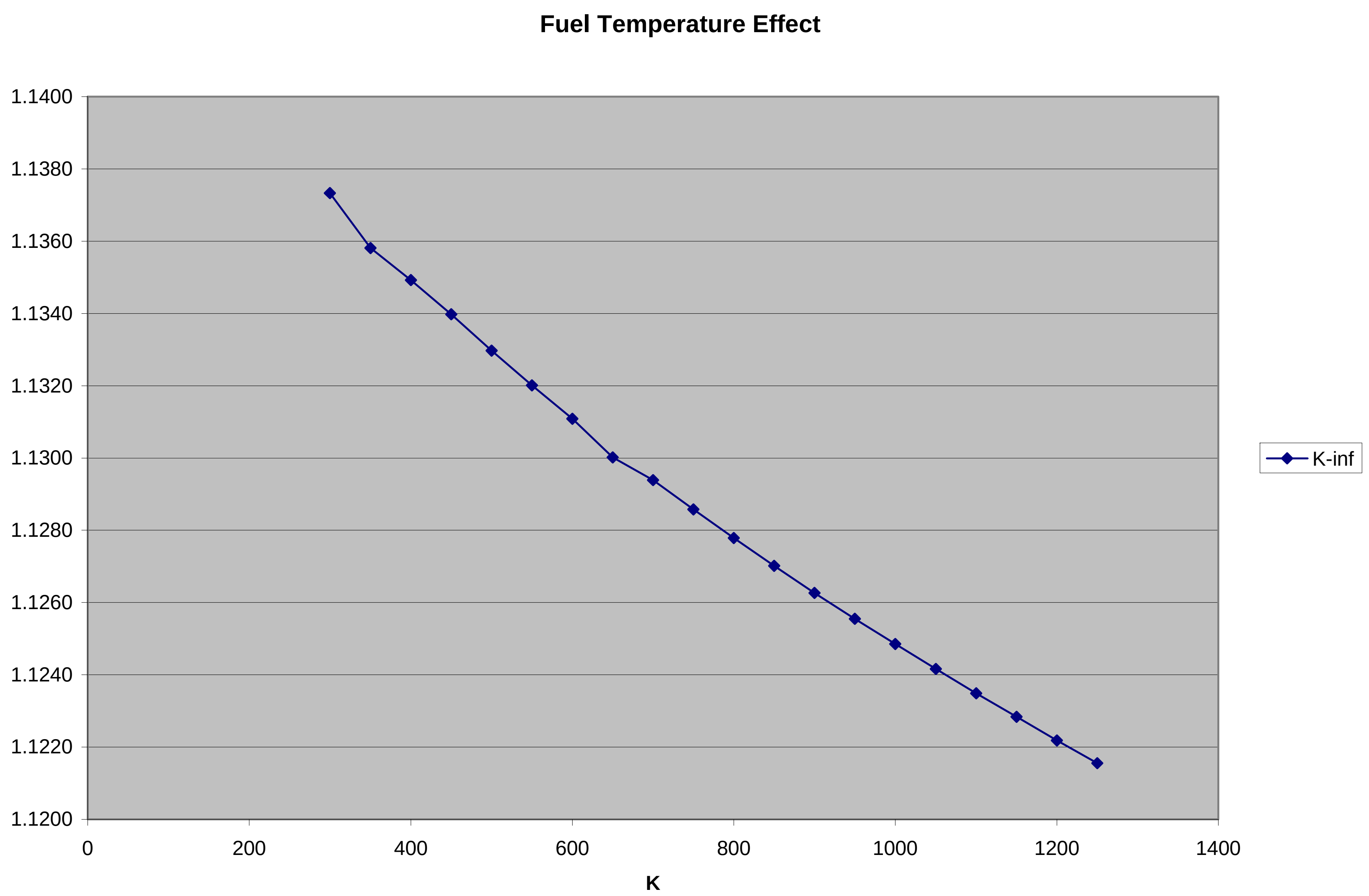


Explanation of the Doppler Broadening of Resonances

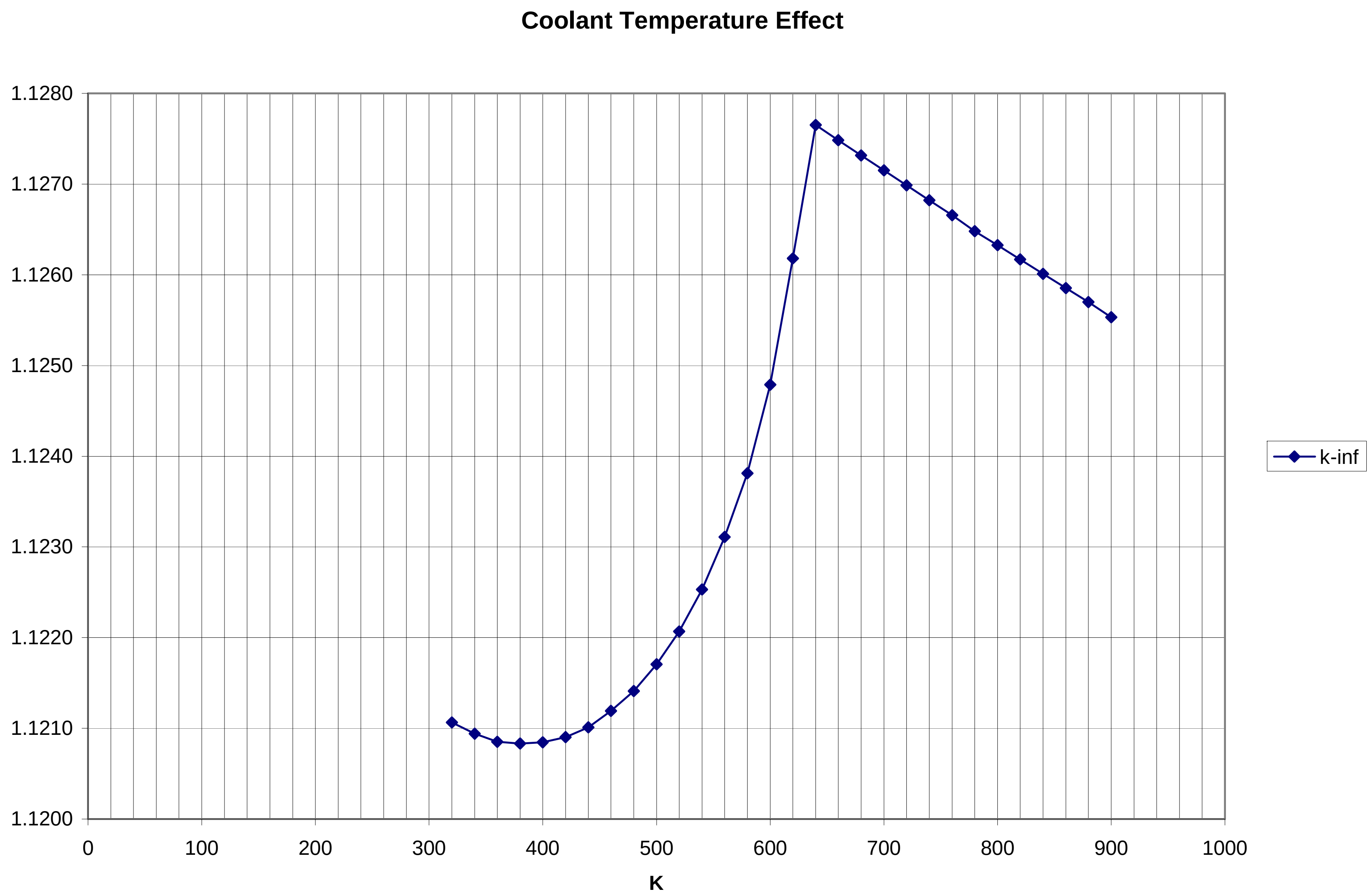
- The cross section depends on the **relative** speed of the neutron with respect to the nucleus



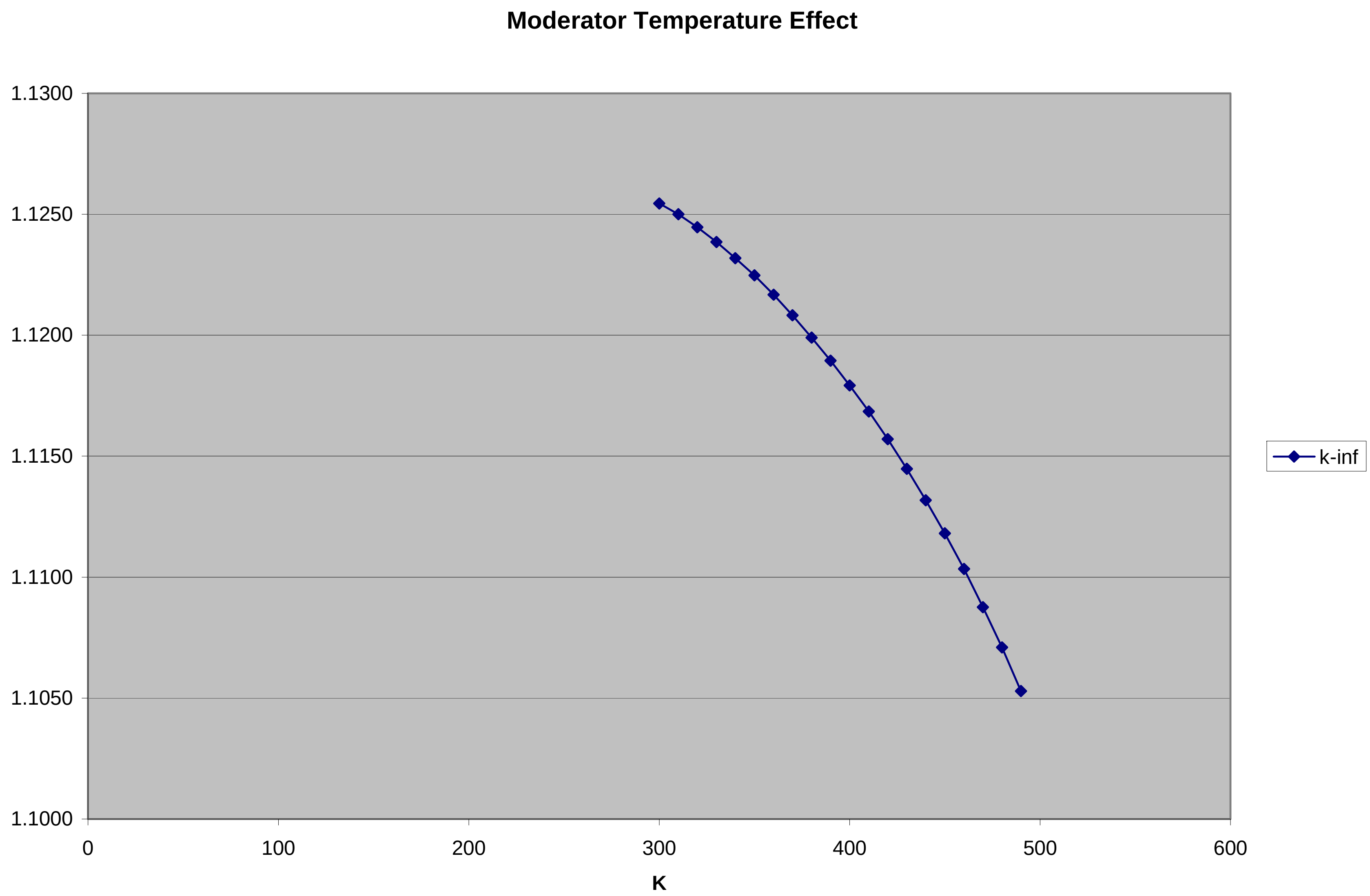
- As nuclei move, neutrons with energy outside the resonance range get to have a relative velocity corresponding to the resonance range.

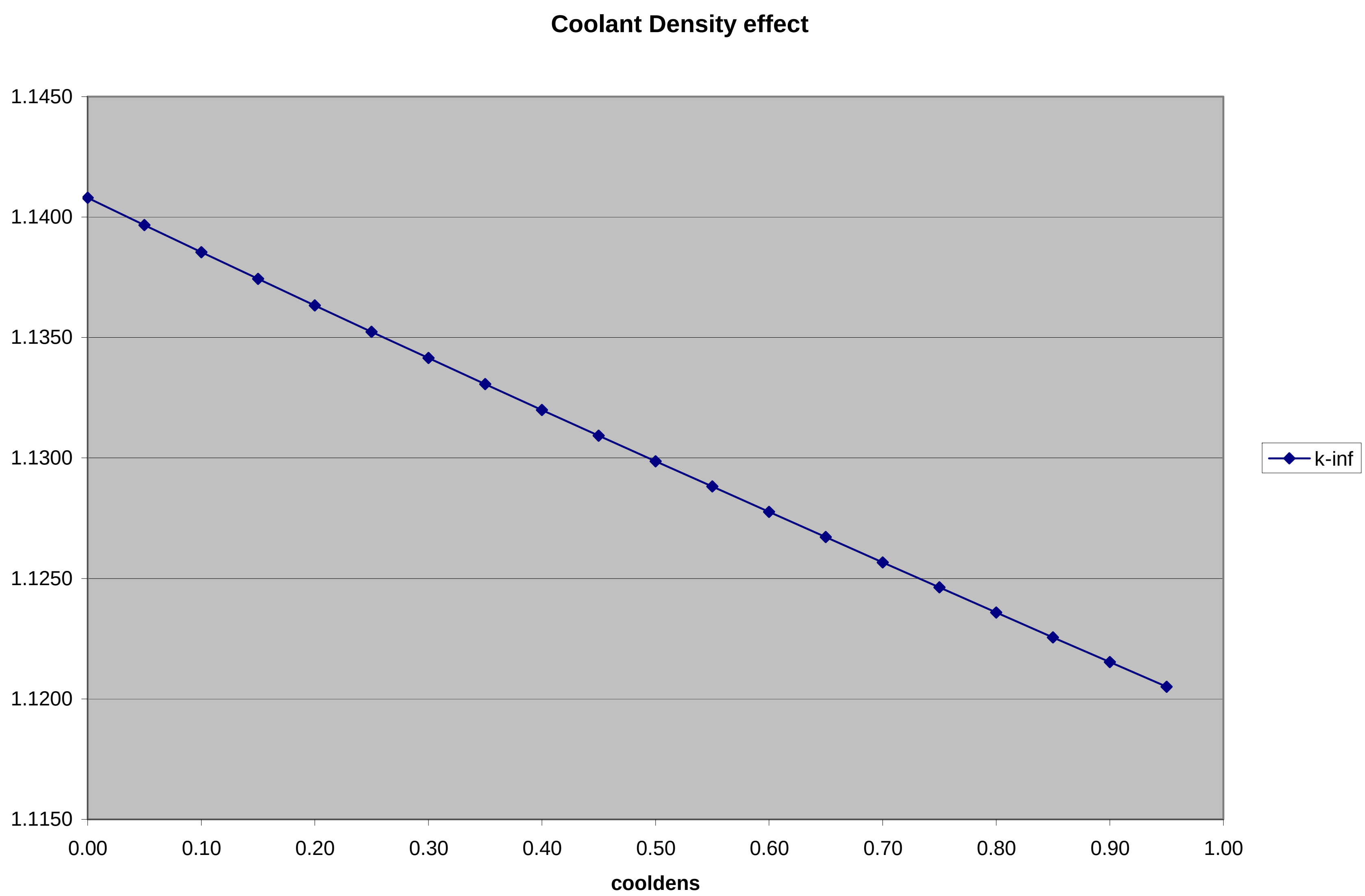


Fuel Temperature Effect
1.1400
1.1380
1.1360
1.1340
1.1320
1.1300
1.1280
1.1260
1.1240
1.1220
1.1200
0
200
400
600
800
1000
1200
1400
K
K-inf



Coolant Temperature Effect
1.1280
1.1270
1.1260
1.1250
1.1240
1.1230
1.1220
1.1210
1.1200
0
100
200
300
400
500
600
700
800
900
1000
K
k-inf


Moderator Temperature Effect
1.1300
1.1250
1.1200
1.1150
1.1100
1.1050
1.1000
0
100
200
300
400
500
600
K
k-inf



Coolant Density effect
1.1450
1.1400
1.1350
1.1300
1.1250
1.1200
1.1150
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
cooldens
k-inf