HEAT CONDUCTION EQUATIONS

5-3. THE HEAT-CONDUCTION EQUATIONS

Figure 5-1 shows a stationary cartesian volume $\Delta x \Delta y \Delta z$ within a heat generating and conductive element. A heat balance on this volume during an increment of time $\Delta\theta$ can be written as*:

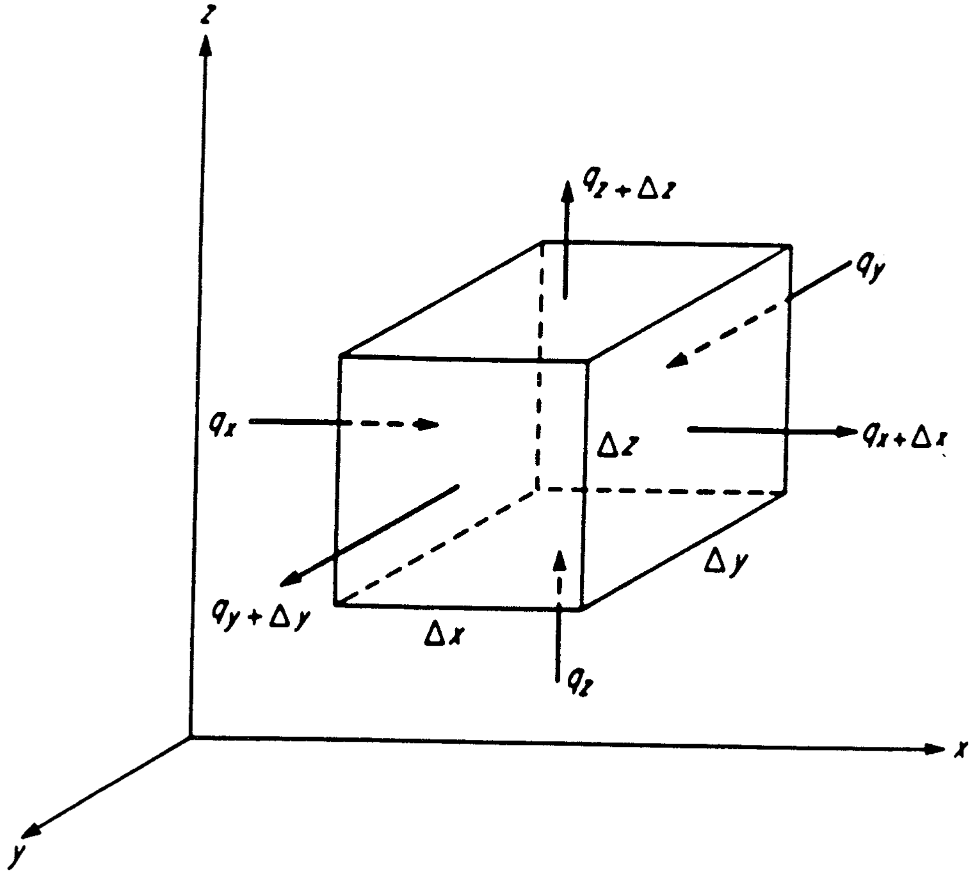

FIG. 5-1. Three-dimensional heat conduction in cartesian coordinates.

$$\begin{pmatrix}\text{Change in internal energy of}\\ \text{material within volume during } \Delta\theta\end{pmatrix} = \begin{pmatrix}\text{Heat conducted into}\\ \text{volume during } \Delta\theta\end{pmatrix} - \begin{pmatrix}\text{Heat conduced}\\ \text{out during } \Delta\theta\end{pmatrix} + \begin{pmatrix}\text{Heat generated}\\ \text{during } \Delta\theta\end{pmatrix}$$

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* Note the similarity to the derivation of the neutron conservation equation, Sec. 3-3.

or

$$(U^{\theta+\Delta\theta} - U^{\theta}) = (q_x + q_y + q_z)\,\Delta\theta - (q_{x+\Delta x} + q_{y+\Delta y} + q_{z+\Delta z})\,\Delta\theta + \Delta x\Delta y\Delta z\, q'''\Delta\theta \tag{5-1}$$

The symbols U^{θ} and $U^{\theta+\Delta\theta}$ are the internal energies of the material within the volume at time θ and $\theta + \Delta\theta$ respectively, and q''' is the volumetric thermal source strength, Btu/hr ft^3. The internal energies can be written as

$$U^{\theta} = \Delta x\Delta y\Delta z\rho u^{\theta} \tag{5-2}$$

and

$$U^{\theta+\Delta\theta} = \Delta x\Delta y\Delta z\rho u^{\theta+\Delta\theta}$$

$$= \Delta x\Delta y\Delta z\left(\rho u^{\theta} + \frac{\partial \rho u^{\theta}}{\partial\theta}\,\Delta\theta\right) \tag{5-3}$$

where ρ is the density of the material of the element, lb$_m$/ft^3 and u is the specific internal energy Btu/lb$_m$. Since $\partial u/\partial\theta = \partial\,(ct)/\partial\theta$, where c is the specific heat of the element, Btu/lb$_m$°F, the above two equations can be combined, for constant ρ and c into the form

$$U^{\theta+\Delta\theta} - U^{\theta} = \Delta x\Delta y\Delta z\rho c\left(\frac{\partial t}{\partial\theta}\right)\Delta\theta \tag{5-4}$$

The symbols q_x and $q_{x+\Delta x}$ are the rates of heat conducted in the x direction, perpendicular to area $\Delta y\Delta z$, at faces x and $x + \Delta x$ respectively, Btu/hr. They can be written, with the help of the *Fourier* equation

$$q_x = -\,kA\,\frac{\partial t}{\partial x} \tag{5-5}$$

as

$$q_x = -\,k\Delta y\Delta z\,\frac{\partial t}{\partial x} \tag{5-6}$$

and

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x}\,\Delta x$$

$$= -\,k\Delta y\Delta z\,\frac{\partial t}{\partial x} - \Delta y\Delta z\,\frac{\partial}{\partial x}\left(k\,\frac{\partial t}{\partial x}\right)\Delta x \tag{5-7}$$

and for constant thermal conductivity k

$$= -\,k\Delta y\Delta z\,\frac{\partial t}{\partial x} - k\Delta y\Delta z\left(\frac{\partial^2 t}{\partial x^2}\right)\Delta x \tag{5-8}$$

Thus

$$q_x - q_{x+\Delta x} = \Delta x\Delta y\Delta z k\left(\frac{\partial^2 t}{\partial x^2}\right) \tag{5-9}$$

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$$q_x = -kA \frac{\partial t}{\partial x} \tag{5-5}$$

as

$$q_x = -k \Delta y \Delta z \frac{\partial t}{\partial x} \tag{5-6}$$

and

$$\begin{aligned} q_{x+\Delta x} &= q_x + \frac{\partial q_x}{\partial x} \Delta x \\ &= -k \Delta y \Delta z \frac{\partial t}{\partial x} - \Delta y \Delta z \frac{\partial}{\partial x}\left(k \frac{\partial t}{\partial x}\right) \Delta x \end{aligned} \tag{5-7}$$

and for constant thermal conductivity k

$$= -k \Delta y \Delta z \frac{\partial t}{\partial x} - k \Delta y \Delta z \left(\frac{\partial^2 t}{\partial x^2}\right) \Delta x \tag{5-8}$$

Thus

$$q_x - q_{x+\Delta x} = \Delta x \Delta y \Delta z k \left(\frac{\partial^2 t}{\partial x^2}\right) \tag{5-9}$$

Writing similar expressions for the heat conducted in the y and z directions, substituting with Eq. 5-4 into Eq. 5-1 and rearranging give

$$\begin{aligned} \Delta x \Delta y \Delta z k \left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right) \Delta \theta + \Delta x \Delta y \Delta z q''' \Delta \theta \\ = \Delta x \Delta y \Delta z \rho c \left(\frac{\partial t}{\partial \theta}\right) \Delta \theta \end{aligned} \tag{5-10}$$

Dividing the entire equation by $\Delta x \Delta y \Delta z \Delta \theta$ and noting that $k/\rho c = \alpha$, the *thermal diffusivity,* ft²/hr, the above equation reduces to the *general heat-conduction equation in cartesian coordinates:*

$$\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right) + \frac{q'''}{k} = \frac{1}{\alpha} \frac{\partial t}{\partial \theta} \tag{5-11}$$

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Putting

$$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \tag{5-12}$$

where $\nabla^2 t$ above is the *Laplacian operator* of temperature in *cartesian coordinates*. Substituting into Eq. 5-11 results in the *general heat-conduction* equation:

$$\nabla^2 t + \frac{q'''}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \theta} \tag{5-13}$$

The special case of steady-state heat conduction, where $\partial t/\partial \theta = 0$, the general equation becomes the *Poisson* equation:

$$\nabla^2 t + \frac{q'''}{k} = 0 \tag{5-14}$$

The special case of no heat generation, where $q''' = 0$, gives the *Fourier equation**

$$\nabla^2 t = \frac{1}{\alpha}\frac{\partial t}{\partial \theta} \tag{5-15}$$

The special case of steady state and no heat generation gives the *Laplace* equation

$$\nabla^2 t = 0 \tag{5-16}$$

Another equation of interest is the *Helmholtz* equation†:

$$\nabla^2 t + B^2 t = 0 \tag{5-17}$$

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* Equations 5-5 and 5-15 share the name of Fourier.

† Recall the reactor equation, Sec. 3-3.

TABLE 5-3a
Differential Equations of Temperature in Heat Conduction

Equation name	Conduction Mode	Equation
General conduction	Transient with heat generation	$\nabla^2 t + \frac{q'''}{k} = \frac{1}{\alpha}\frac{\partial t}{\partial \theta}$
Poisson	Steady state with heat generation	$\nabla^2 t + \frac{q'''}{k} = 0$
Fourier	Transient with no heat generation	$\nabla^2 t = \frac{1}{\alpha}\frac{\partial t}{\partial \theta}$
Laplace	Steady state with no heat generation	$\nabla^2 t = 0$
Helmholtz	Steady state with linear function of temperature term	$\nabla^2 t + B^2 t = 0$

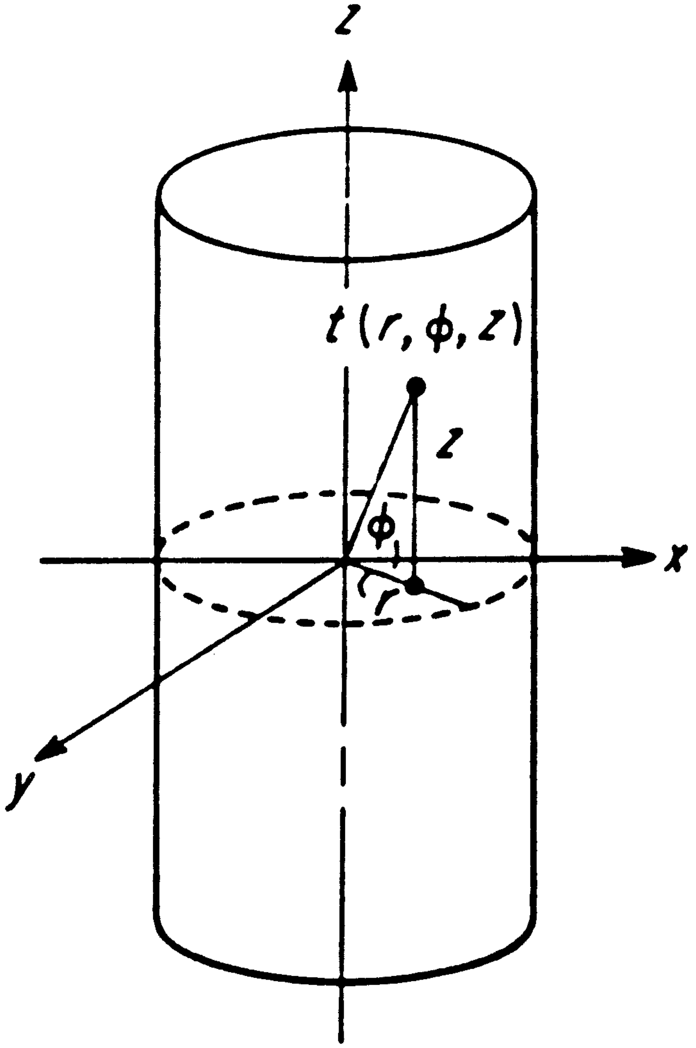

FIG. 5-2. Cylindrical coordinate system.

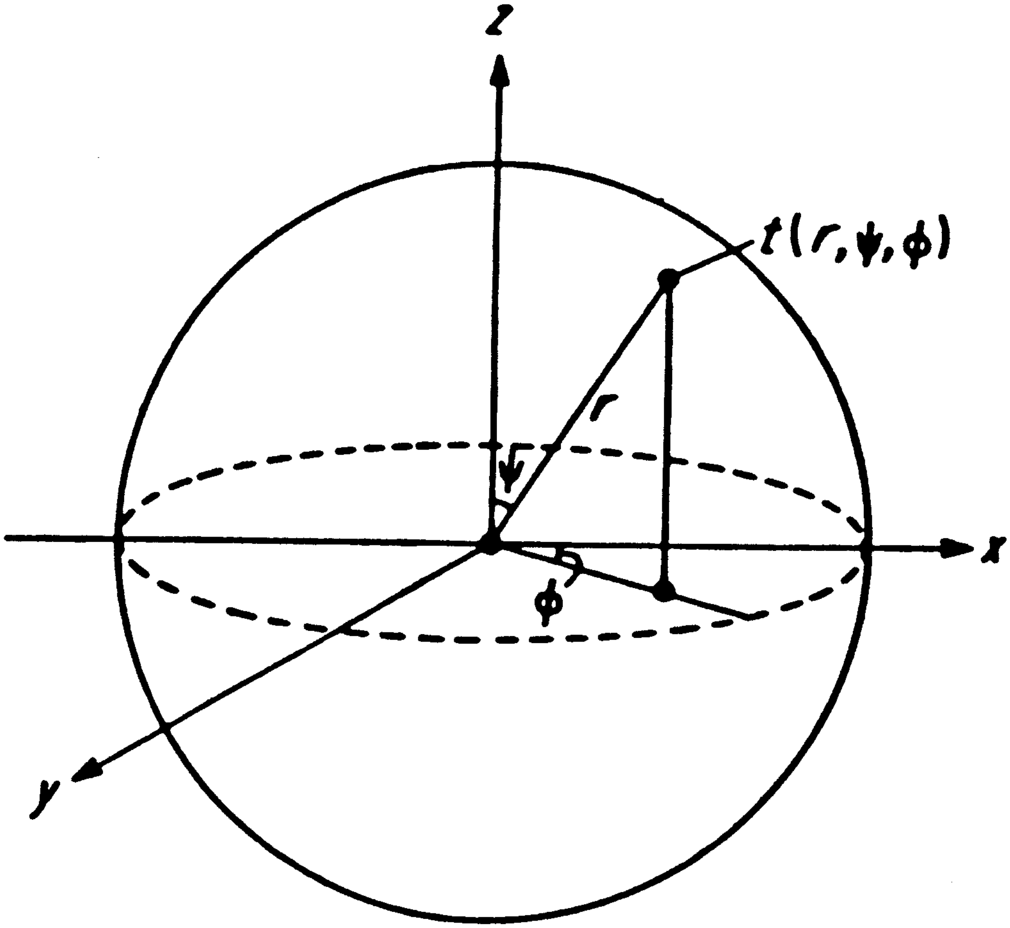

FIG. 5-3. Spherical coordinate system.

TABLE 5-3b
The Laplacian of Temperature, $\nabla^2 t$

Coordinates	Cartesian	Cylindrical	Spherical
3-dimensional	$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$	$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}$ $+ \frac{1}{r^2}\frac{\partial^2 t}{\partial \varphi^2} + \frac{\partial^2 t}{\partial z^2}$	$\frac{\partial^2 t}{\partial r^2} + \frac{2}{r}\frac{\partial t}{\partial r}$ $+ \frac{1}{r^2 \tan \psi}\frac{\partial t}{\partial \psi}$ $+ \frac{1}{r^2}\frac{\partial^2 t}{\partial \psi^2}$ $+ \frac{1}{r^2 \sin^2 \psi}\frac{\partial^2 t}{\partial \varphi^2}$
1-dimensional (in x or r)	$\frac{d^2 t}{dx^2}$	$\frac{d^2 t}{dr^2} + \frac{1}{r}\frac{dt}{dr}$	$\frac{d^2 t}{dr^2} + \frac{2}{r}\frac{dt}{dr}$

FUEL ROD HEAT CONDUCTION

PRIME ASSUMPTION

ONE DIMENSIONAL STEADY STATE CONDUCTION

MAIN APPLICATIONS

- FLAT FUEL PLATES
- CYLINDRICAL FUEL RODS
- SPHERICAL FUEL PELLETS

ADDITIONAL REFERENCE

"HEAT TRANSFER" 7TH EDITION J. P. HOLMAN

GENERAL EQUATION :

HEAT FLOW RATE $\propto$ NORMAL TEMPERATURE GRADIENT

$$\frac{q_x}{A} \propto -\frac{\partial T}{\partial x}$$

INCLUDING A PROPORTIONALITY CONSTANT

$$q_x = -kA\frac{\partial T}{\partial x}$$

$$q_x = -kA\frac{dT}{dx} \quad \text{(ONE DIMENSIONAL)}$$

HEAT CONDUCTION EQUATIONS

DERIVATION OF HEAT TRANSFER EQUATIONS IN THREE DIMENSIONS GIVES THE FOLLOWING:

CARTESIAN CO-ORDINATES

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{K} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

CYLINDRICAL CO-ORDINATES

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{K} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

SPHERICAL CO-ORDINATES

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \tan\psi}\frac{\partial T}{\partial \psi} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \psi} + \frac{1}{r^2 \sin^2\psi}\frac{\partial^2 T}{\partial \psi^2} + \frac{q'''}{K} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

WHERE :

q''' = HEAT GENERATED $(J/s\,m^2)$

k = THERMAL CONDUCTIVITY $(J/s\,m\,°c)$

ρ = DENSITY (kg/m^3)

c = SPECIFIC HEAT $(J/kg\,°c)$

α = THERMAL DIFFUSIVITY (m^2/s)

$\alpha = k/\rho c$

HEAT CONDUCTION EQUATIONS

FOR STEADY STATE CONDITIONS RIGHT HAND SIDE OF EACH EQUATION IS ZERO

FOR ONE DIMENSIONAL APPLICATION WITH HEAT GENERATION EQUATIONS ARE :

<u>CARTESIAN CO-ORDINATES</u> (INFINITE SLAB)

$$\frac{d^2T}{dx^2} + \frac{q'''}{k} = 0$$ (TEXT BOOK 5-22

<u>CYLINDRICAL CO-ORDINATES</u> (INFINITE ROD)

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{q'''}{k} = 0$$ (TEXT BOOK 5-41)

<u>SPHERICAL CO-ORDINATES</u> (UNIFORM SPHERE)

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} + \frac{q'''}{k} = 0$$ (TEXT BOOK 5-64)