Control Systems

Part 3: Laplace Transforms

Learning objectives

- To state the definition of Laplace transform
- To be able to use Laplace transform table to solve differential equations
- To examine different performance measures in time domain
- To represent system in terms of transfer functions using Laplace transforms.

Differential equation and operator representations

The linear form of this model is:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = b_{n-1}\frac{d^{n-1}}{dt^{n-1}}u(t) + \ldots + b_0 u(t)$$

Introducing a differential operator $\rho\langle\circ\rangle$:

$$\rho\langle f(t)\rangle = \rho f(t) \triangleq \frac{df(t)}{dt}$$

$$\rho^n\langle f(t)\rangle = \rho^n f(t) = \rho\left\langle \rho^{n-1}\langle f(t)\rangle\right\rangle = \frac{df^n(t)}{dt^n}$$

Then

$$\rho^n y(t) + a_{n-1}\rho^{n-1}y(t) + \ldots + a_0 y(t) = b_{n-1}\rho^{n-1}u(t) + \ldots + b_0 u(t)$$

Definition of Laplace transform

Consider a continuous time variable $y(t)$; $0 \le t < \infty$. The Laplace transform pair associated with $y(t)$ is defined as

$$\mathcal{L}\left[y(t)\right] = Y(s) = \int_{0^-}^{\infty} e^{-st} y(t) dt$$

$$\mathcal{L}^{-1}\left[y(s)\right] = y(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} e^{st} Y(s) ds$$

Laplace transform of a derivative term

The Laplace transform of the derivative of a function:

$$\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0^-)$$

where $y(0^-)$ is the initial condition associated with $y(t)$.

Laplace transform table

$f(t) \quad (t \geq 0)$	$\mathcal{L}[f(t)]$	Region of Convergence
1	$\frac{1}{s}$	$\sigma > 0$
$\delta_D(t)$	1	$\lvert\sigma\rvert < \infty$
t	$\frac{1}{s^2}$	$\sigma > 0$
$t^n \quad n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$e^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{s-\alpha}$	$\sigma > \Re\{\alpha\}$
$te^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{(s-\alpha)^2}$	$\sigma > \Re\{\alpha\}$
$\cos(\omega_o t)$	$\frac{s}{s^2+\omega_o^2}$	$\sigma > 0$
$\sin(\omega_o t)$	$\frac{\omega_o}{s^2+\omega_o^2}$	$\sigma > 0$
$e^{\alpha t}\sin(\omega_o t + \beta)$	$\frac{(\sin\beta)s + \omega_o^2\cos\beta - \alpha\sin\beta}{(s-\alpha)^2+\omega_o^2}$	$\sigma > \Re\{\alpha\}$
$t\sin(\omega_o t)$	$\frac{2\omega_o s}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
$t\cos(\omega_o t)$	$\frac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
$\mu(t) - \mu(t-\tau)$	$\frac{1-e^{-s\tau}}{s}$	$\lvert\sigma\rvert < \infty$

Properties of Laplace transform

$f(t)$	$\mathcal{L}[f(t)]$	Names
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(s)$	Linear combination
$\frac{dy(t)}{dt}$	$sY(s) - y(0^-)$	Derivative Law
$\frac{d^k y(t)}{dt^k}$	$s^k Y(s) - \sum_{i=1}^{k} s^{k-i} \left. \frac{d^{i-1}y(t)}{dt^{i-1}} \right\|_{t=0^-}$	High order derivative
$\int_{0^-}^{t} y(\tau)d\tau$	$\frac{1}{s}Y(s)$	Integral Law
$y(t-\tau)\mu(t-\tau)$	$e^{-s\tau}Y(s)$	Delay
$ty(t)$	$-\frac{dY(s)}{ds}$	
$t^k y(t)$	$(-1)^k \frac{d^k Y(s)}{ds^k}$	
$\int_{0^-}^{t} f_1(\tau) f_2(t-\tau)d\tau$	$F_1(s)F_2(s)$	Convolution
$\lim_{t\to\infty} y(t)$	$\lim_{s\to 0} sY(s)$	Final Value Theorem
$\lim_{t\to 0^+} y(t)$	$\lim_{s\to\infty} sY(s)$	Initial Value Theorem
$f_1(t)f_2(t)$	$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F_1(\zeta)F_2(s-\zeta)d\zeta$	Time domain product
$e^{at}f_1(t)$	$F_1(s-a)$	Frequency Shift

Definition of transfer functions

Taking Laplace Transforms converts the differential equation into the following algebraic equation

$$s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \dots + a_0 Y(s)$$

$$= b_{n-1} s^{n-1} U(s) + \dots + b_0 U(s) + f(s; x_o)$$

This can be expressed as

$$Y(s) = G(s)U(s)$$

$G(s)$ is called the *transfer function*.

$$G(s) = \frac{B(s)}{A(s)}$$

$$A(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$$

$$B(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0$$

Transfer functions for state space models

Taking Laplace transform in the state space model equations yields

$$sX(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}X(s) + \mathbf{D}U(s)$$

and hence

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}x(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$
$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0)$$

$$Y(s) = \mathbf{G}(s)U(s)$$
$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$G(s)$ is the system transfer function.

Laplace transform of time delay

Often practical systems have a time delay between input and output. This is usually associated with the transport of material from one point to another. For example, it takes time for the coolant to flow from the reactor to the steam generator,then this will invariably introduce a delay.

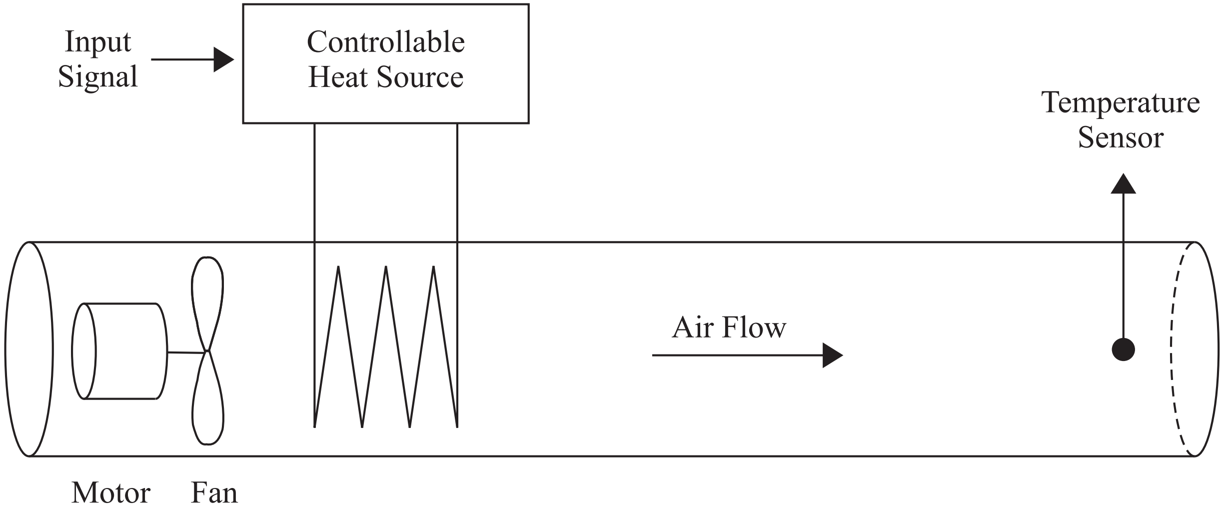
The transfer function of a pure delay is of the form (see Table):

$$H(s) = e^{-sT_d}$$

where T_d is the delay (in seconds). T_d will typically vary depending on the transportation speed.

Example of heating system

As a simple example, the following heating system has a pure time delay:



Transfer function of the heating system

The transfer function from input (the voltage applied to the heating element) to the output (the temperature as seen by the thermocouple) is approximately of the form:

$$H(s) = \frac{Ke^{-sT_d}}{(\tau s + 1)}$$

Transfer functions describe the input-output properties of linear systems in algebraic form

(as oppose to a differential form before taking Laplace Transform)

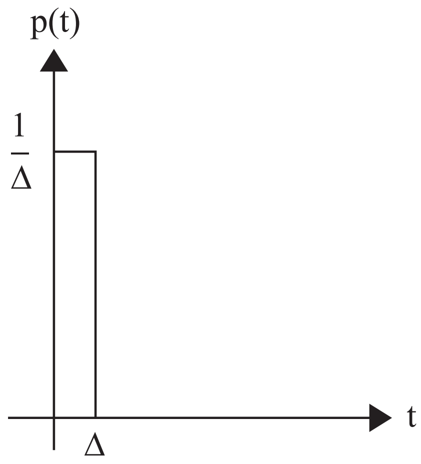
Stability

A system is stable if any bounded input produces a bounded output for all bounded initial conditions.

Impulse and step responses of linear systems

The transfer function of a continuous time system is the Laplace transform of its response to an impulse (Dirac's delta) with zero initial conditions.

The impulse function can be thought of as the limit ($\Delta \to 0$) of the pulse shown below.



Steady state step response

The steady state response (provided it exists) for a unit step is given by

$$\lim_{t \to \infty} y(t) = y_\infty = \lim_{s \to \infty} sG(s)\frac{1}{s} = G(0)$$

where $G(s)$ is the transfer function of the system.

Performance measures

Steady state value, y_∞: the final value of the step response (this is meaningless if the system has poles in the CRHP).

Rise time, t_r: The time elapsed up to the instant at which the step response reaches, for the first time, the value $k_r y_\infty$. The constant k_r varies from author to author, being usually either 0.9 or 1.

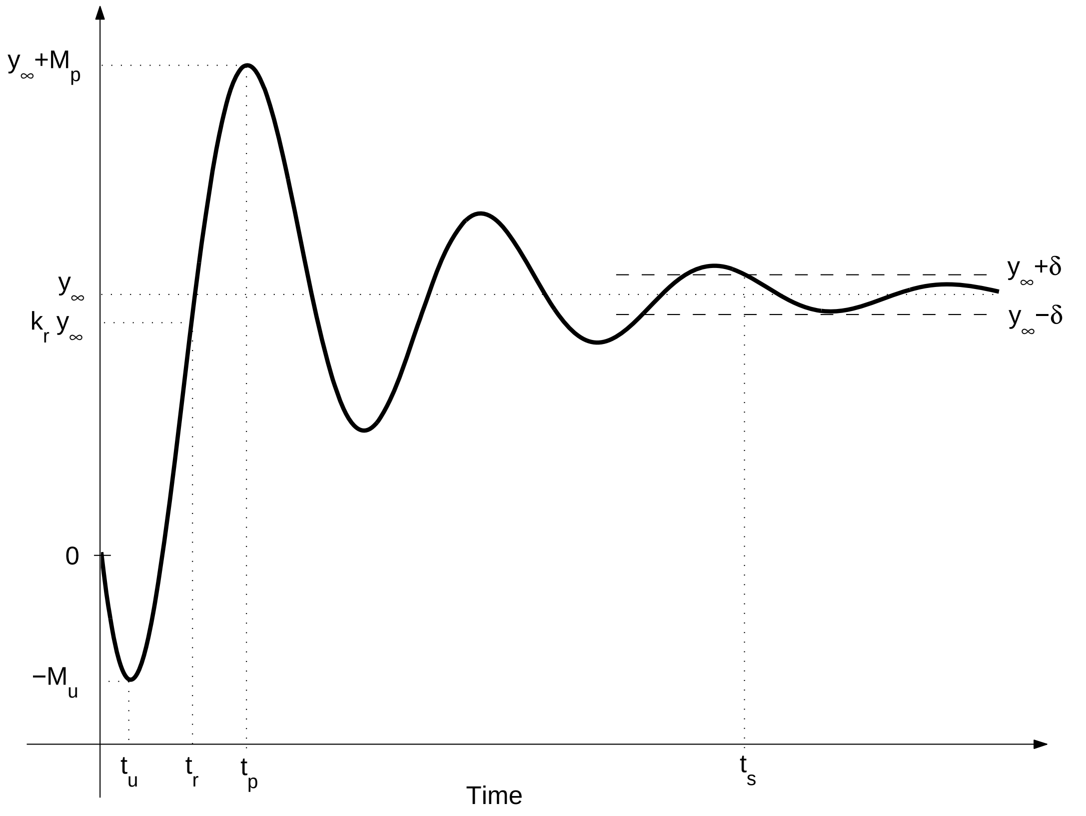
Overshoot, M_p: The maximum instantaneous amount by which the step response exceeds its final value. It is usually expressed as a percentage of $\boldsymbol{y}_\infty$

Performance measures

Undershoot, M_u: the (absolute value of the) maximum instantaneous amount by which the step response falls below zero.

Settling time, t_s: the time elapsed until the step response enters (without leaving it afterwards) a specified deviation band, $\pm\delta$, around the final value. This deviation δ, is usually defined as a percentage of y_∞, say 2% to 5%.

Step response indicators



Poles, zeros and time responses

We will consider a general transfer function of the form

$$H(s) = K\frac{\prod_{i=1}^{m}(s-\beta_i)}{\prod_{l=1}^{n}(s-\alpha_l)}$$

β_1, β_1,..., β_m and α_1, α_2, ,,, α_n are the zeros and poles of the transfer function, respectively. The relative degree is $n_r \overset{\Delta}{=} n - m$.

Poles

Any scalar rational transfer function can be expanded into a partial fraction expansion, each term of which contains either a single real pole, a complex conjugate pair or multiple combinations with repeated poles.

The stability requires that the poles have strictly negative real parts, i.e., they need to be in the open left half plane (OLHP) of the complex plane s. This implies that, for continuous time systems, the stability boundary is the imaginary axis.

First order pole

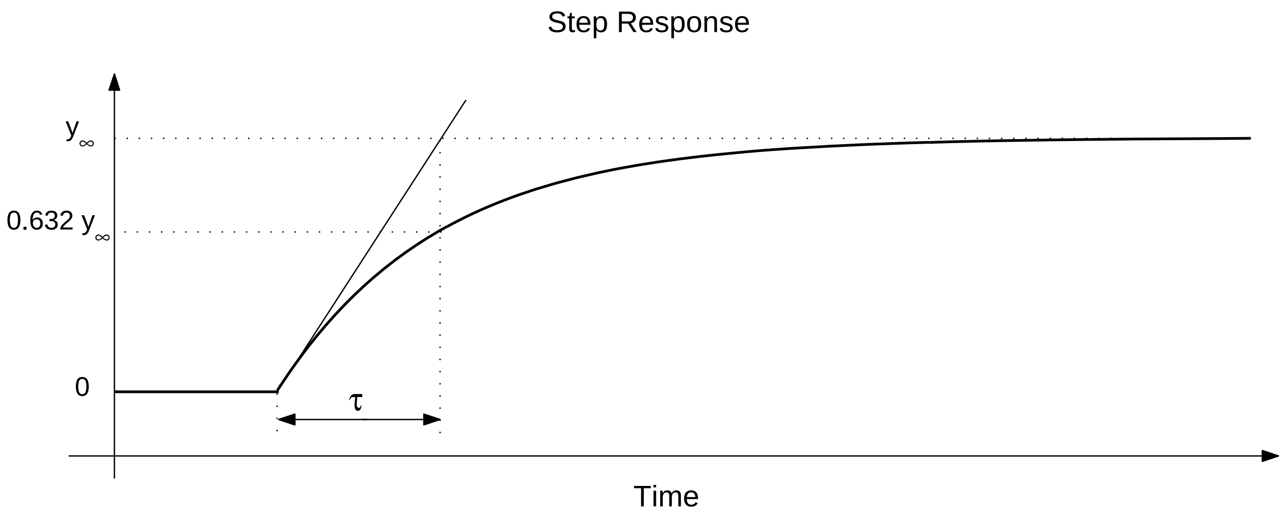
A general first order pole contributes

$$H_1(s) = \frac{K}{\tau s + 1}$$

The response of this system to a unit step can be computed as

$$y(t) = \mathcal{L}^{-1}\left[\frac{K}{s(\tau s + 1)}\right] = \mathcal{L}^{-1}\left[\frac{K}{s} - \frac{K\tau}{\tau s + 1}\right] = K(1 - e^{-\frac{t}{\tau}})$$

Step response of a first order system- time constant



A complex conjugate pair

For the case of a pair of complex conjugate poles, it is customary to study a *canonical second order system* having the transfer function.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\psi\omega_n s + \omega_n^2}$$

Step Response of canonical second order system

The Laplace transform can be represented as

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s+\psi\omega_n}{(s+\psi\omega_n)^2+\omega_d^2} - \frac{\psi\omega_n}{(s+\psi\omega_n)^2+\omega_d^2} \\ &= \frac{1}{s} - \frac{1}{\sqrt{1-\psi^2}}\left[\sqrt{1-\psi^2}\frac{s+\psi\omega_n}{(s+\psi\omega_n)^2+\omega_d^2} - \psi\frac{\omega_d}{(s+\psi\omega_n)^2+\omega_d^2}\right] \end{aligned}$$

On applying the inverse Laplace transform, it follows:

$$y(t) = 1 - \frac{e^{-\psi\omega_n t}}{\sqrt{1-\psi^2}}\sin(\omega_d t + \beta)$$

Frequency response

We next study the system response to a rather special input, namely a sine wave. The reason for doing so is that the response to sine waves also contains rich information about the response to other signals.

Let the transfer function be

$$H(s) = K\frac{\sum_{i=0}^{m} b_i s^i}{s^n + \sum_{k=1}^{n-1} a_k s^k}$$

Then the steady state response to the input $\sin(wt)$ is

$$y(t) = |H(jw)|\sin(wt + \phi(w))$$

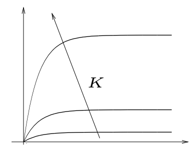
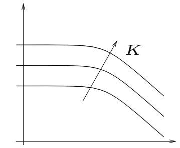
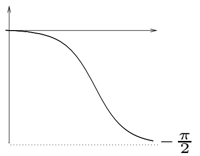
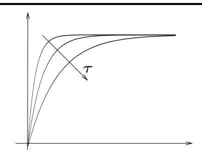
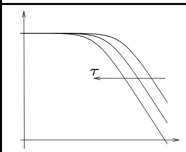
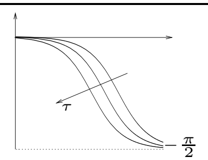
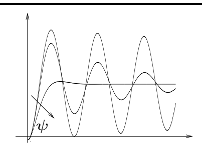
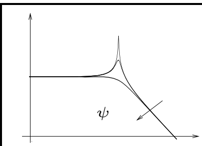
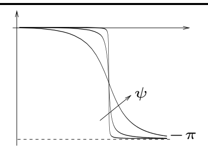
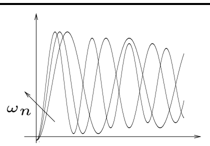
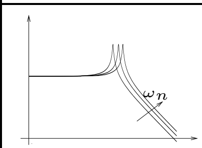
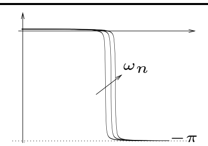
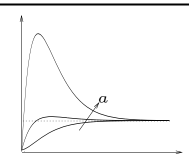
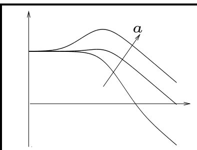
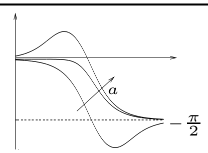
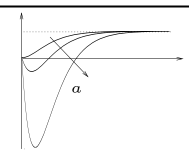
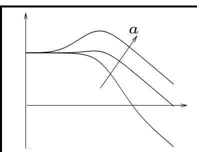
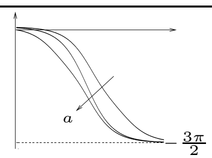
where

$$H(j\omega) = |H(j\omega)|e^{j\phi(\omega)}$$

In summary

A sine wave input forces a sine wave at the output with the same frequency. Moreover, the amplitude of the output sine wave is modified by a factor equal to the magnitude of H(jw) and the phase is shifted by a quantity equal to the phase of H(jw).

System models and influence of parameter variations

System	Parameter	Step response	Bode (gain)	Bode(phase)
$\frac{K}{\tau s+1}$	K			
	τ			
$\frac{\omega_n^2}{s^2+2\psi\omega_n s+\omega^2}$	ψ			
	ω_n			
$\frac{as+1}{(s+1)^2}$	a			
$\frac{-as+1}{(s+1)^2}$	a			

Summary

- There are two key approaches to linear dynamic models:
  - time domain, and
  - frequency domain
- Although these two approaches are largely equivalent, they each have their own particular advantages and it is therefore important to have a good grasp of each.

Time domain

- In the time domain
  - systems are modeled by differential equations
  - systems are characterized by the evolution of their variables (output etc.) in time
  - the evolution of variables in time is computed by solving differential equations

Frequency domain

- In the frequency domain,
  - modeling exploits the key linear system property that the steady state response to a sinusoid is again a sinusoid of the same frequency; the system only changes amplitude and phase of the input in a fashion uniquely determined by the system at that frequency,
  - systems are modeled by transfer functions, which capture this impact as a function of frequency.

Stability

- With respect to the important characteristic of stability, a continuous time system is
  - stable if and only if the real parts of all poles are strictly negative
  - marginally stable if at least one pole is strictly imaginary and no pole has strictly positive real part
  - unstable if the real part of at least one pole is strictly positive