Control Systems

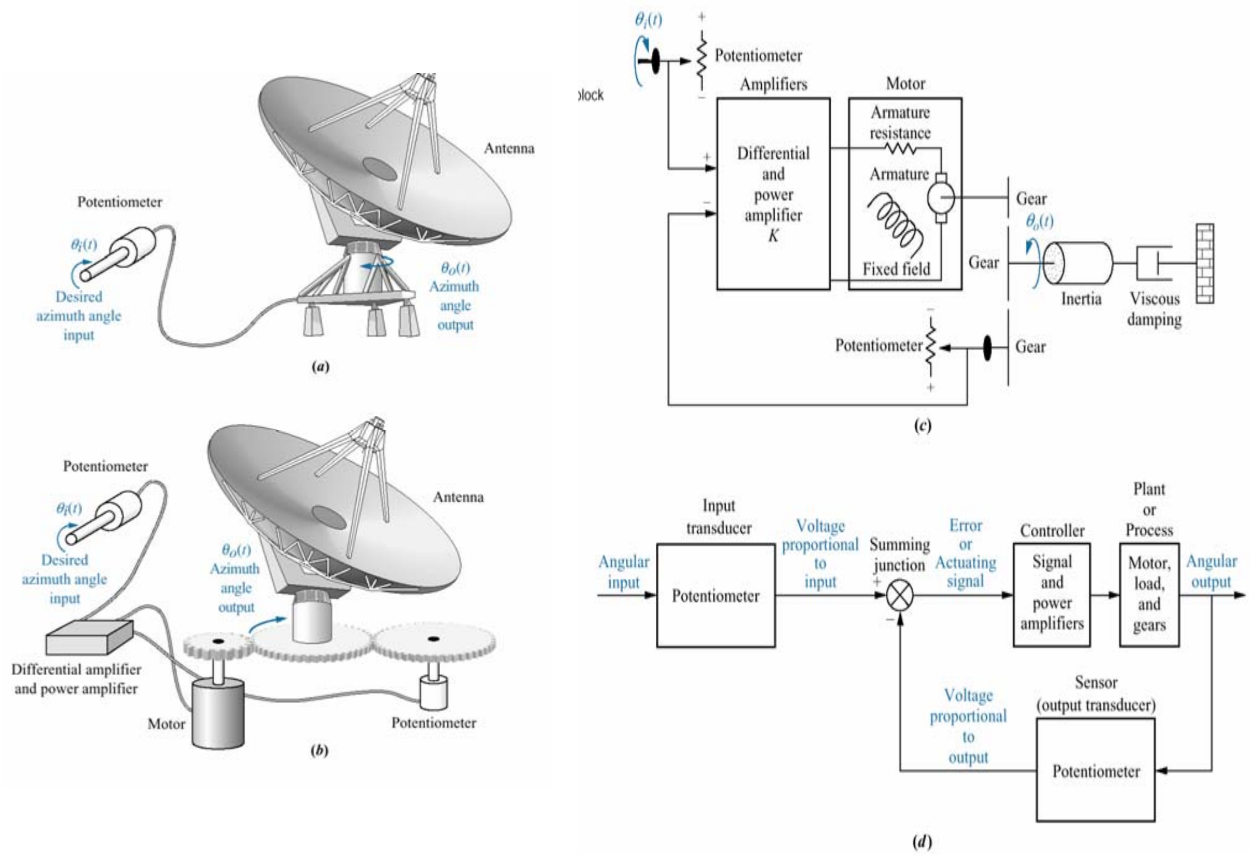
Part 2: Modeling of Dynamic Systems

Learning objectives

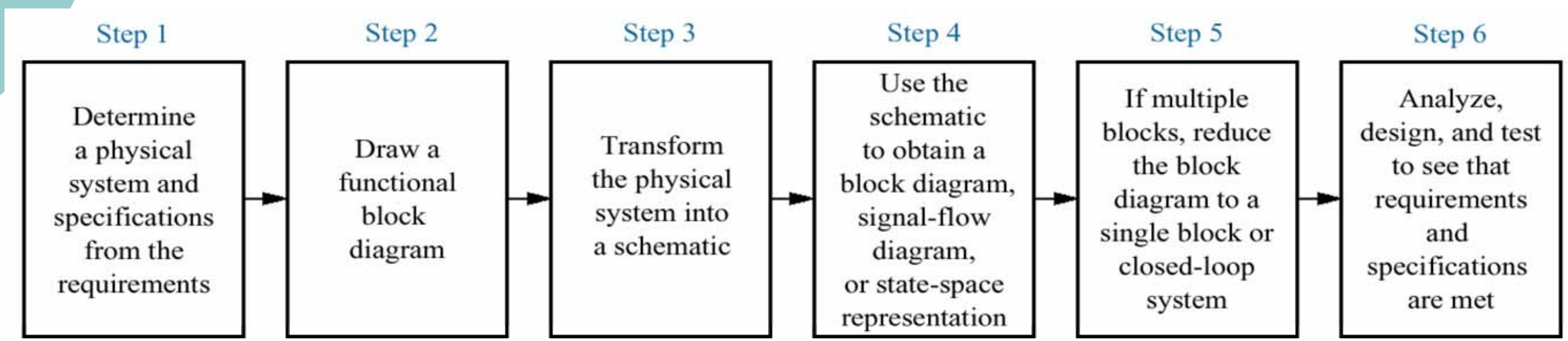
- To derive mathematical representation of physical systems
- To solve differential equations for time domain responses
- To convert the differential equations to another domain, i.e. Laplace Transforms
- To represent system in terms of transfer functions using Laplace transforms.

Mathematical representation of the system

(a) system concept;
(b) detailed layout;
(c) schematic;
(d) functional block diagram



Process of modeling



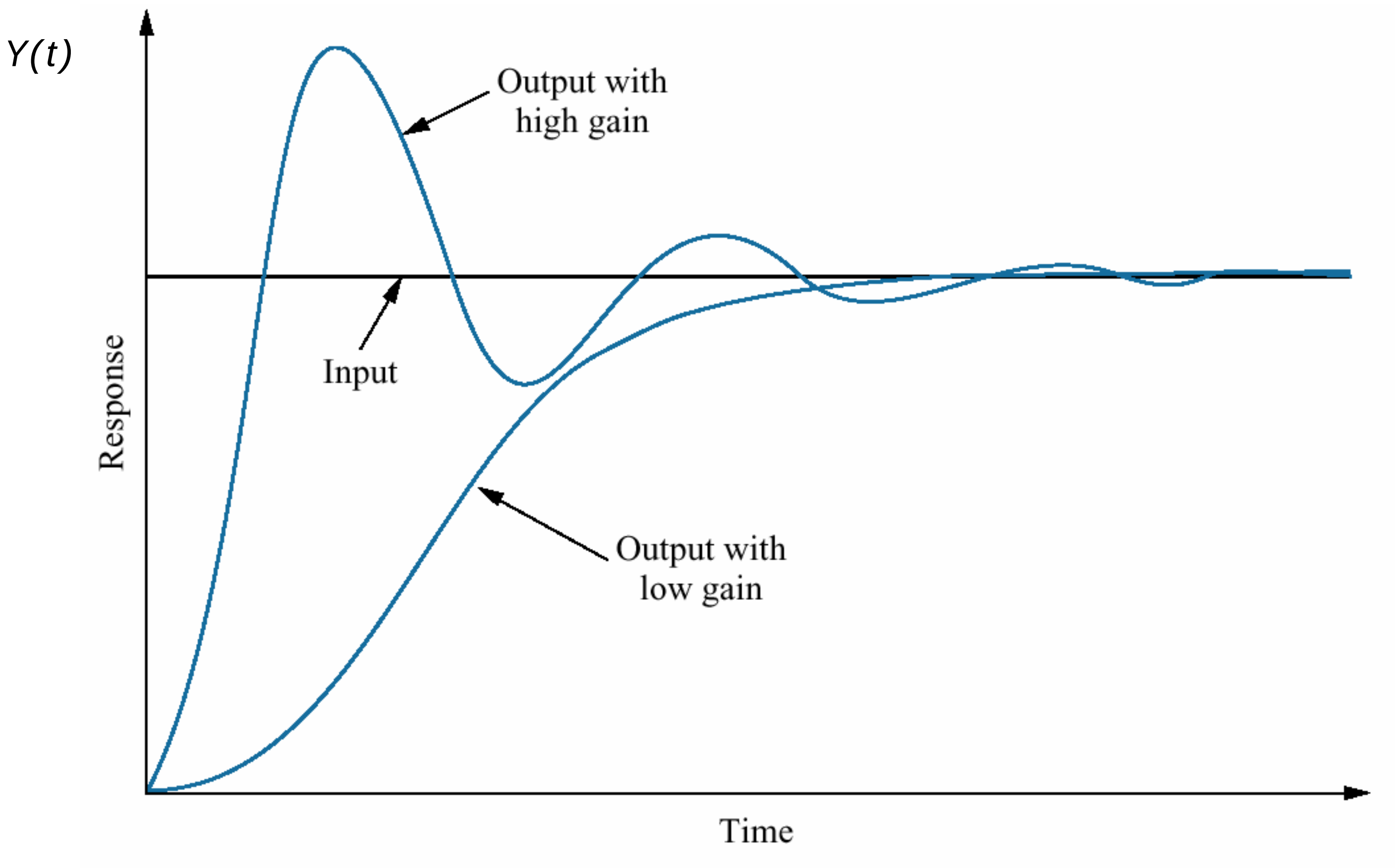
Differential equation representation of physical systems

The underlying physical principles will lead to the formation of differential equation representation of the system as follows:

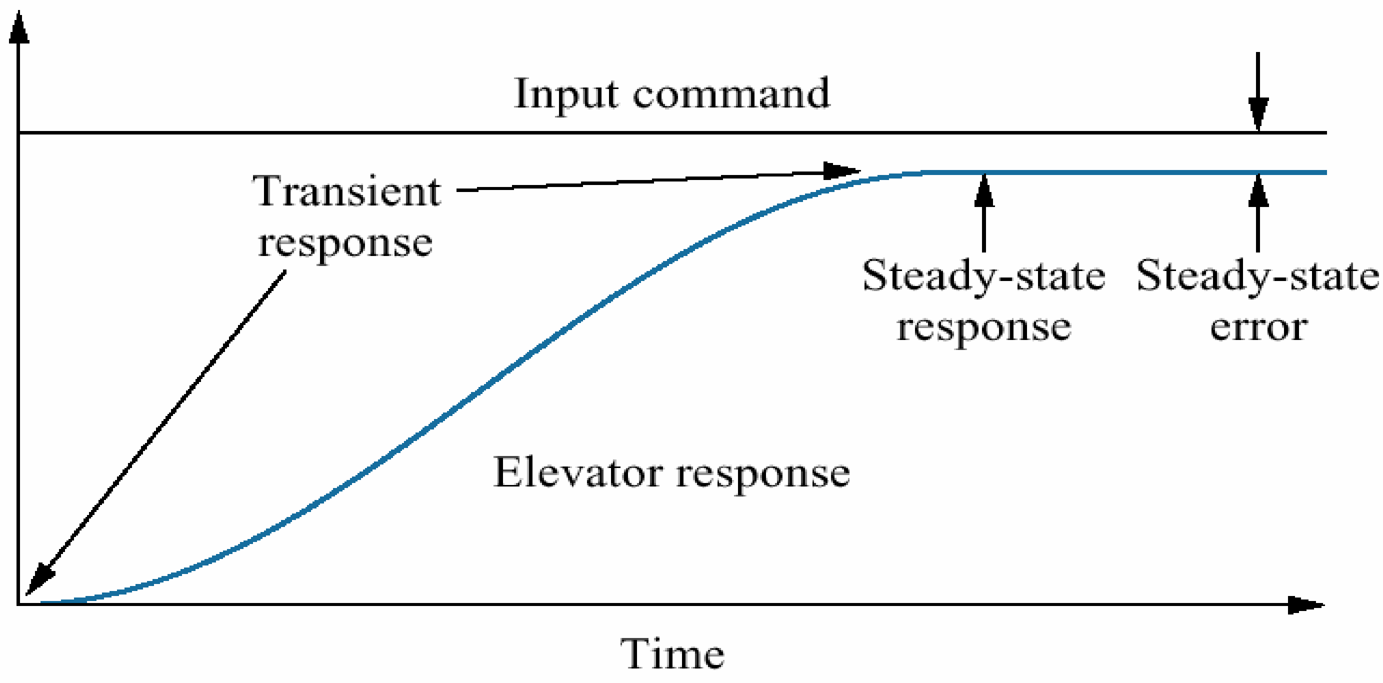
$$a_n \cdot \frac{d^n}{dt^n} y + a_{n-1} \cdot \frac{d^{n-1}}{dt^{n-1}} y + \dots + a_1 \cdot \frac{d}{dt} y + a_0 \cdot y = u(t)$$

The idea is to find out how system output, y(t), responds to a particular type of input excitation, u(t).

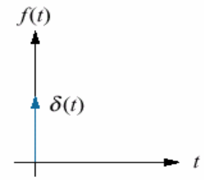
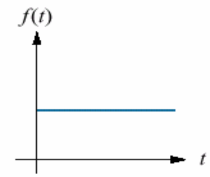
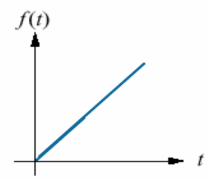
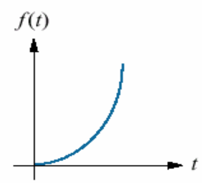
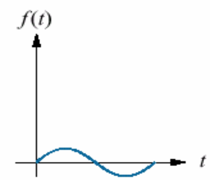
Typical step response



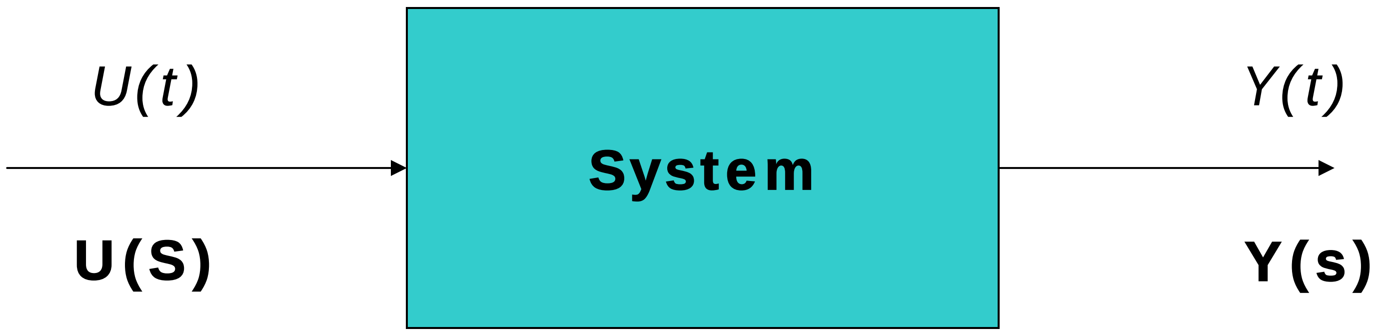
Some definition of terms



Typical test inputs for control system analysis

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)\,dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

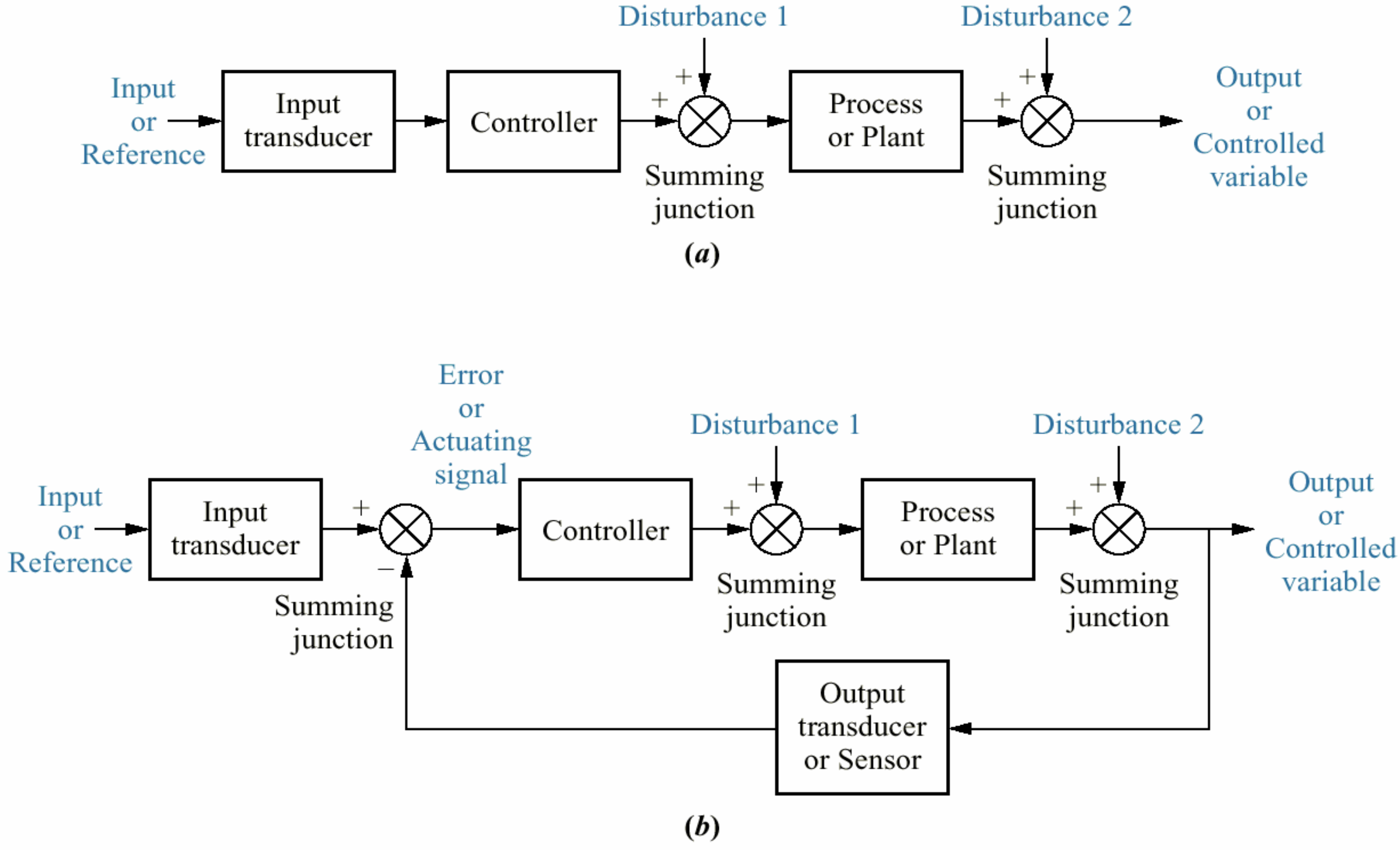
Transfer function representation of the system



Transfer Function: G(s) = Y(s) / U(s)

Under zero initial conditions

A typical control system



State space representation of dynamic systems

In addition to the transfer function representation, one can also represent a dynamic system in terms of state space form

$$\frac{dx}{dt} = f(x(t), u(t), t)$$

$$y(t) = g(x(t), u(t), t)$$

State space representation of linear systems

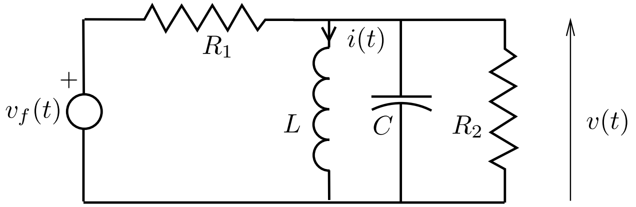
If the system is linear, one can have the following representation:

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t)$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

Example

Consider the simple electrical network shown below. Assume we want to derive the dynamic relationships between the input $v_f(t)$ the voltage $v(t)$



On applying fundamental network laws we obtain the following equations:

$$v(t) = L\frac{di(t)}{dt}$$

$$\frac{v_f(t) - v(t)}{R_1} = i(t) + C\frac{dv(t)}{dt} + \frac{v(t)}{R_2}$$

Example (Cont)

These equations can be rearranged as follows:

$$\frac{di(t)}{dt} = \frac{1}{L}v(t)$$

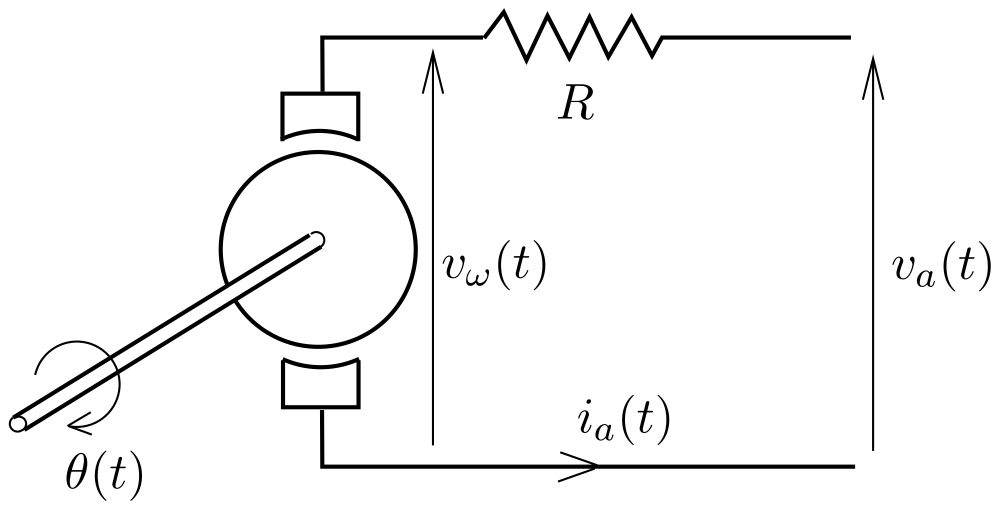
$$\frac{dv(t)}{dt} = -\frac{1}{C}i(t) - \left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)v(t) + \frac{1}{R_1C}v_f(t)$$

We have a linear state space model with

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right) \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{R_1C} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}; \quad \mathbf{D} = \mathbf{0}$$

Example of a DC motor

Consider a separately excited D.C. motor. Let $v_a(t)$ denote the armature voltage, $\theta(t)$ the output angle. A simplified schematic diagram of this system is shown below



Parameters of DC motor

Let

J	- be the inertia of the shaft
$\tau_e(t)$	- the electrical torque
$i_a(t)$	- the armature current
$k_1;\ k_2$	- constants
R	- the armature resistance

Application of well known principles of physics tells us that the various variables are related by:

Dynamic model of a DC motor

$$J\ddot{\theta}(t) = \tau_e(t) = k_1 i_a(t)$$

$$v_\omega(t) = k_2\dot{\theta}(t)$$

$$i_a(t) = \frac{v_a(t) - k_2\dot{\theta}(t)}{R}$$

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-k_1 k_2}{R} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_1}{R} \end{bmatrix} v_a(t)$$

Summary of modeling of dynamic systems

- In order to systematically design a controller for a particular system, one needs a formal - though possibly simple - description of the system. Such a description is called a model.
- A model is a set of mathematical equations that are intended to capture the effect of certain system variables on certain other system variables.

Points to note

- *Certain system variables*: It is usually neither possible nor necessary to model the effect of every variable on every other variable; one therefore limits oneself to certain subsets. Typical examples include the effect of input on output, the effect of disturbances on output, the effect of a reference signal change on the control signal, or the effect of various unmeasured internal system variables on each other.

Points to note

- *Capture*: A model is never perfect and it is therefore always associated with a modeling error. The word capture highlights the existence of errors, but does not yet concern itself with the precise definition of their type and effect.
- *Intended*: This word is a reminder that one does not always succeed in finding a model with the desired accuracy and hence some iterative refinement may be needed.
- *Set of mathematical equations*: There are numerous ways of describing the system behavior, such as linear or nonlinear differential or difference equations.

Points to note

- Models are classified according to properties of the equation they are based on. Examples of classification include:

Model Attribute	*Contrasting Attribute*	*Asserts whether or not …*
Single input Single output	Multiple input multiple output	… the model equations have one input and one output only
Linear	Nonlinear	… the model equations are linear in the system variables
Time varying	Time invariant	… the model parameters are constant
Continuous	Sampled	… model equations describe the behavior at every instant of time, or only in discrete *samples* of time
Input-output	State space	… the model equations rely on functions of input and output variables only, or also include the so called *state variables*.
Lumped parameter	Distributed parameter	… the model equations are ordinary or partial differential equations